Math 212

Last time: $B = I_1 \times \cdots \times I_n$, box

$f : B \to \mathbb{R}$, bounded

$P = P_1 \times \cdots \times P_n$, partition of B

$m_J(f) = \inf f(J)$, $M_J(f) = \sup f(J)$

$L(f,P) = \sum_J m_J(f)\,\mathrm{vol}(J)$, $U(f,P) = \sum_J M_J(f)\,\mathrm{vol}(J)$ lower and upper sums

$\underline{\int}_B f = \sup_P \{L(f,P)\}$, $\overline{\int}_B f = \inf_P \{U(f,P)\}$ lower and upper integrals

Def. f is integrable if $\underline{\int}_B f = \overline{\int}_B f$, and in that case

$$\int_B f = \underline{\int}_B f = \overline{\int}_B f.$$

Prop. If P and Q are any two partitions of B, then $L(f,P) \le U(f,Q)$. [Every lower sum is less than or equal to every upper sum.]

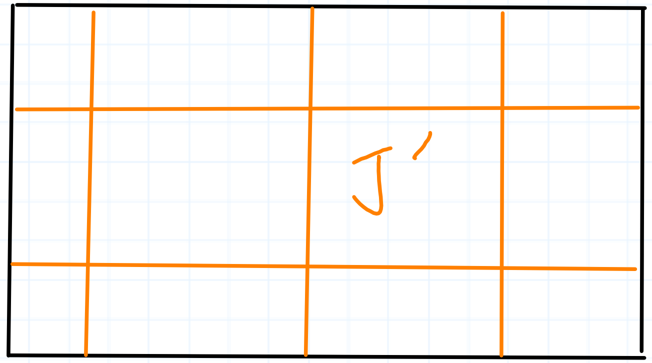
Pf/ Step 1. If P is any partition, then $L(f,P) \le U(f,P)$.

Reason: $m_J(f) = \inf f(J) \le \sup f(J) = M_J(f) \Longrightarrow$

$$L(f,P) = \sum_J m_J(f)\,\mathrm{vol}(J) \le \sum_J M_J(f)\,\mathrm{vol}(J) = U(f,P).$$

Step 2. If P, P' are partitions of B and $P' \supseteq P$ (i.e., P' refines P), then $L(f,P) \le L(f,P')$ and $U(f,P') \le U(f,P)$. [Refining a partition makes the lower and upper sums "better".]

Reason: If P' refines P, then each subbox J for P is divided into (possibly several) subboxes of P':



J

[$\mathrm{Box}(P') :=$ subboxes for P']

$$J = \bigcup_{\substack{J' \in \mathrm{Box}(P') \\ J' \subseteq J}} J'$$

If $J' \subseteq J$, then $m_{J'}(f) = \inf f(J') \geq m_J(f) = \inf f(J)$.

Consider the lower sums for each partition:

$$L(f,P) = \sum_{J \in \mathrm{Box}(P)} m_J(f)\,\mathrm{vol}(J), \quad L(f,P') = \sum_{J' \in \mathrm{Box}(P')} m_{J'}(f)\,\mathrm{vol}(J').$$

Group the summands of $L(f,P')$ according subboxes for P:

$$L(f,P') = \sum_{J \in \mathrm{Box}(P)} \sum_{\substack{J' \in \mathrm{Box}(P') \\ J' \subseteq J}} m_{J'}(f)\, \mathrm{vol}(J')$$

and compare:

exercise

$$m_J(f,P)\, \mathrm{vol}(J) = m_J(f,P) \left[\sum_{\substack{J' \in \mathrm{Box}(P') \\ J' \subseteq J}} \mathrm{vol}(J') \right]$$

$$= \sum_{J' \subseteq J} m_J(f,P)\, \mathrm{vol}(J')$$

$$\leq \sum_{J' \subseteq J} m_{J'}(f,P)\, \mathrm{vol}(J')$$

Summing over $J \in \mathrm{Box}(P)$ shows $L(f,P) \leq L(f,P')$.

The argument that $U(f,P') \le U(f,P)$ is similar.

Step 3. Let P, Q be any two partitions of B. Then there exists a common refinement T.

Reason: Say $P = P_1 \times \cdots \times P_n$ and $Q = Q_1 \times \cdots \times Q_n$ where P_i and Q_i are partitions of I_i $\forall i$. Define T by

$$T = (P_1 \cup Q_1) \times \cdots \times (P_n \cup Q_n).$$

Then $T \supseteq P$ & $T \supseteq Q$.

Step 4. Given partitions P, Q, let T be their common refinement. Then

$$L(f,P) \overset{\text{step 2}}{\le} L(f,T) \overset{\text{step 1}}{\le} U(f,T) \overset{\text{step 3}}{\le} U(f,Q). \quad \square$$

Corollary $\underline{\int}_B f$ and $\overline{\int}_B f$ exist.

Pf/ Fix a partition Q. We have seen that $L(f,P) \le U(f,Q)$ for all partitions P. Thus, the set of all lower sums is bounded above (e.g., by any particular upper sum). By completeness*, $\sup_P \{L(f,P)\} = \underline{\int}_B f$ exists. The argument that $\overline{\int}_B f$ exists is similar. $\square$

* **Completeness:** Every nonempty subset of $\mathbb{R}$ that is bounded above has a sup. (Equivalently, every nonempty subset of $\mathbb{R}$ that is bounded below has an inf.)