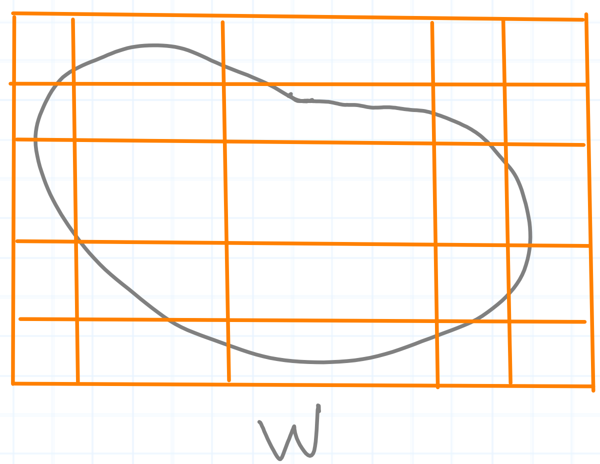
Math 212

Integration

General motivation. Let $f : \mathbb{R}^n \to \mathbb{R}$ denote density (mass/volume). To compute the mass of $W \subseteq \mathbb{R}^n$, cover W by rectangles:



For each rectangle R, pick a point $p_R \in R$. Modify f by forcing it to be zero outside of W.

$$\tilde{f} : \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$p \longmapsto \begin{cases} 0 & \text{if } p \notin W \\ f(w) & \text{if } p \in W. \end{cases}$$

Then, the total mass of W is by definition $\int_W f$ (to be defined below)

and is estimated by the Riemann sum:

$$\text{mass}(W) = \int_W f \approx \sum_R \tilde{f}(p_R)\,\text{vol}(R)$$

where the volume of R, denoted $\text{vol}(R)$ is the product of the lengths of the sides of R.

Rigorous definition of the integral.

A nonempty, compact interval in $\mathbb{R}$ is a set of the form

$$I = \{ x \in \mathbb{R} : a \le x \le b \}$$

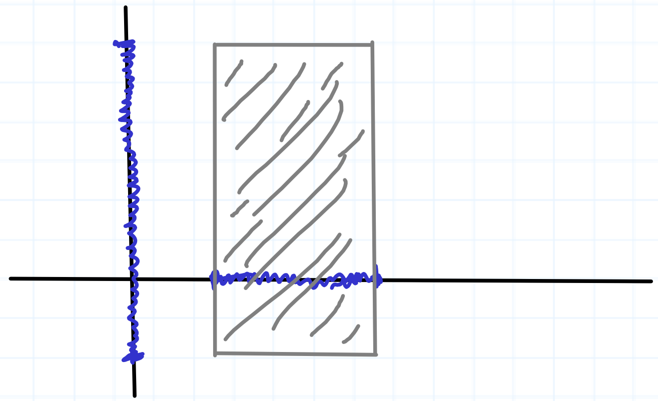
for some $a \le b$ in $\mathbb{R}$.

A **box** in $\mathbb{R}^n$ is a Cartesian product of n nonempty compact intervals:

$$B = I_1 \times \cdots \times I_n .$$

Example

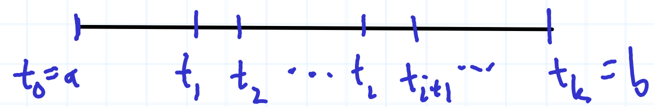
$$B = [1,2] \times [-1,3]$$



A **partition** of an interval $I = \{ x : a \le x \le b \}$ is a collection of points

$$P = \{ t_0, \ldots, t_k \}$$

such that $a = t_0 < t_1 < \cdots < t_k = b$.



The **subintervals** of P are the sets $[t_{i-1}, t_i] = \{ x \in \mathbb{R} : t_{i-1} \le x \le t_i \}$, A partition of a box $B = I_1 \times \cdots \times I_n \subseteq \mathbb{R}^n$ is a Cartesian product of partitions of the I_i:

$$P = P_1 \times \cdots \times P_n, \qquad P_i \text{ a partition of } I_i.$$

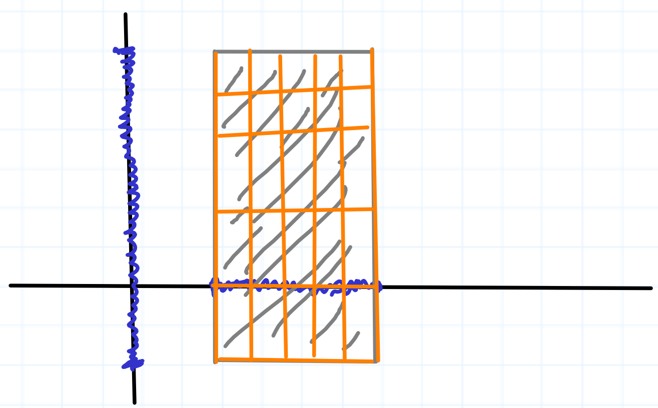
Example

$$B = \underbrace{[1,2]}_{I_1} \times \underbrace{[-1,3]}_{I_2},$$

$$P_1 = \{1, 1.2, 1.4, 1.6, 1.8, 2\}$$

$$P_2 = \{-1, 0, 1, 2, 2.5, 3\}$$

$$P = P_1 \times P_2 = \{(1,-1), (1,0), (1,1), \ldots, (2,2.5), (2,3)\}$$



The elements of P are the intersections of the horizontal and vertical lines.

A partition $P = P_1 \times \cdots \times P_n$ of a box B divides B into subboxes:

$$B_{i_1, \ldots, i_n} = [t_{i_1}, t_{i_1+1}] \times \cdots \times [t_{i_n}, t_{i_n+1}].$$

The volume of a compact interval $I = [a,b]$ is its length: $b-a$.

The volume of a box $B = I_1 \times \cdots \times I_n$ is the product of the lengths of its sides

$$\mathrm{vol}(B) = \prod_{i=1}^{n} \mathrm{vol}(I_i).$$

Fix a box $B = I_1 \times \cdots \times I_n$ and a partition $P = P_1 \times \cdots \times P_n$ of B.

Let $f : B \to \mathbb{R}$ be a bounded function, i.e., there exists $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in B$.

For each subbox J of P, define

$$m_J(f) = \inf\{f(x) : x \in J\} = \inf f(J)$$

(inf = infimum = greatest lower bound = glb)

$$M_J(f) = \sup\{f(x) : x \in J\} = \sup f(J)$$

(sup = supremum = least upper bound = lub).

Define the lower sum for f with respect to P:

$$L(f,P) := \sum_J m_J(f)\,\mathrm{vol}(J),$$

the sum being over all subboxes J of P.

Define the upper sum for f with respect to P:

$$U(f,P) = \sum_J M_J(f)\,\mathrm{vol}(J).$$

Define the lower and upper integrals of f over B by

$$\underline{\int}_B f = \sup_P L(f,P), \qquad \overline{\int}_B f = \inf_P U(f,P)$$

the sup and inf taken as P varies over all partitions of B.

Def. If $\underline{\int}_B f = \overline{\int}_B f$, then f is integrable and

$$\int_B f = \underline{\int}_B f = \overline{\int}_B f.$$