Math 212

* See course webpage.

Course Summary

1. What is n-dimensional volume? (Riemann integral for $\mathbb{R}^n$.)

Example Define $B_n = \{ x \in \mathbb{R}^n : |x| \leq 1 \}$, the unit n-dimensional ball in $\mathbb{R}^n$. Let $v_n = \text{vol}(B_n)$ [to be defined later]. Then

n	0	1	2	3	4	5	6	7	8	...
v_n	1	2	3.14	4.10	4.93	5.26	5.18	4.71	4.05	

2. How do you calculate it in closed form? (Fubini's thm. and the change of variables thm.)

3. How do you integrate a differential form? (Determinants of submatrices of the Jacobian matrix.)

4. Why would you want to?

[• ultimate generalization of the fundamental thm. of calculus:

Stokes' thm. $\nearrow$ Gauss' thm. $\searrow$ Green's thm.

• Calculate flux, flow, and work (vector fields)

• div, grad, curl]

5. Extra topics: • differential geometry (curvature)

(if time) • Maxwell's equations

• differential equations

Big picture

Multivariable Calculus

Study $f: \mathbb{R}^n \to \mathbb{R}^m$ via linearization

Key object: Jacobian matrix

$$Jf(p) = \left(\frac{\partial f_i}{\partial x_j}(p) \right)$$

211
diff'l calc.

212
integral calc.

Study the linear map associated with $Jf(p)$ – the derivative (tangent vectors, rates of change).

Study integrals of determinants of submatrices of $Jf(p)$ (volume, flow, flux).

211 Review question

Let $f(x,y) = (x^2+y,\ x-y,\ x-y^3)$ and let $p = (1,2)$.

(i) What is Df_p ?

Answer:

$$Jf = \begin{bmatrix} 2x & 1 \\ 1 & -1 \\ 1 & -3y^2 \end{bmatrix}, \quad Jf(p) = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 1 & -12 \end{bmatrix}$$

$$Df_p(x,y) = (2x+y,\ x-y,\ x-12y)$$

(ii) What is a reasonable geometric interpretation of Df_p?

Answer: Thinking of f as a parametrized surface, then Df_p parametrizes the tangent plane to the image of f at p, translated back to the origin.