First Pointers.

- A proof consists solely of complete sentences. A sentence starts with a capital letter and ends with a period. Avoid starting a sentence with a mathematical symbol.
- The symbol "$\Rightarrow$" means "implies", as in $x = 3 \Rightarrow 2x = 6$. It does not mean "equals" or "my next thought is", etc. As you are reading what you have written, make sure that substituting the word "implies" for "$\Rightarrow$" makes sense.
- The symbol "$\Leftrightarrow$" means "if and only if" (which is sometimes abbreviated "iff"). If P and Q are statements that are either true or false, then $P \Leftrightarrow Q$ means $P \Rightarrow Q$ and $Q \Rightarrow P$, i.e., the truth of P implies the truth of Q, and conversely, the truth of Q implies that of P.
- The symbol "$\forall$" means "for all" and "$\exists$" means "there exists". I use "s. t." for "such that". I do not use "$\ni$" for "such that", since it conflicts with the following usage: $\{1, 2, 3\} \ni 2$. Also, do *not* use "$\vee$" for "or", "$\wedge$" for "and", or "$\sim$" for "not". It is just as easy to write the words, and it is a lot easier to read the words.
- Only give a proof by contradiction or to prove the contrapositive if a straightforward proof is not as clear.
- When giving a counterexample (disproving) a statement, be as concrete and specific as possible. Try to find the most simple counterexample. In this way, the form your argument should take when disproving a statement is the opposite of that used in providing a proof. A proof requires a general argument, covering all possible cases.
- When writing down a calculation, avoid crossing out terms (for example, when terms cancel in fractions or when they add up to zero). This type of bookkeeping is easy for the writer, who is crossing out sequentially, but is usually confusing for the reader.
- If you use the phrase "by definition" in a proof, make sure to be specific, e.g. "by definition of the derivative . . . ".

Templates. Here are some generic templates for proofs. The symbols P and Q denote mathematical statements that may be true or false. If you do not use these templates, you should have a good excuse.

Theorem 1 *Let S be the set of . . . , and let T be the set of . . .*
Then $S \subseteq T$.

Proof. Let $s \in S$. Then blah, blah, blah, ...

. . . .

. . . .

It follows that $s \in T$. $\square$

Theorem 2 *Let* S *be the set of* ..., *and let* T *be the set of* ...
Then $S = T$.

Proof. Let $s \in S$. Then blah, blah, blah, ...

. . . .

. . . .

It follows that $s \in T$. Hence, $S \subseteq T$.

Now take $t \in T$. It follows that blah, blah, blah, ...

. . . .

. . . .

Thus, $t \in S$. Hence, $T \subseteq S$, as well. $\square$

Theorem 3 $P \Leftrightarrow Q$.

Proof. First, suppose P. Then blah, blah, blah, ...

. . . .

. . . .

It follows that Q.

Conversely, suppose Q. Then blah, blah, blah, ...

. . . .

. . . .

It follows that P. $\square$

Sometimes the two parts of the proof are explicitly labeled, as in the following.

Theorem 4 $P \Leftrightarrow Q$.

Proof. ($\Rightarrow$) Suppose P. Then blah, blah, blah, ...

. . . .

. . . .

It follows that Q.

($\Leftarrow$) Suppose Q. Then blah, blah, blah, ...

. . . .

. . . .

It follows that P. $\square$

Theorem 5 *For all $x, y, z \in S$, it follows that [some statement proposition involving x, y, and z].*

Proof. Let $x, y, z \in S$. Then blah, blah, blah. (Show the proposition holds for this arbitrary choice of elements x, y, and z.) □

A typical induction proof.

Theorem 6

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

for $n = 1, 2, \ldots$

Proof. We will prove this by induction. First note that the statement holds when $n = 1$:

$$1 = \frac{1(1+1)}{2}.$$

Next, suppose the statement holds for some n:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

It follows that

$$\begin{aligned} 1 + 2 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= (n+1)\left(\frac{n}{2} + 1\right) \\ &= \frac{(n+1)(n+2)}{2}, \end{aligned}$$

and the result holds for $n + 1$, too. Hence, the statement holds for all $n = 1, 2, \ldots$ by induction. □

Common proof-writing mistakes.

I. The backwards proof.

Theorem 7 *Suppose $x \in \mathbb{R}$. Then $(x+1)^2 - (x-1)^2 = 4x$.*

Incorrect proof. Calculate:

$$\begin{aligned} (x+1)^2 - (x-1)^2 &= 4x \\ (x^2 + 2x + 1) - (x^2 - 2x + 1) &= 4x \\ x^2 + 2x + 1 - x^2 + 2x - 1 &= 4x \\ 4x &= 4x. \end{aligned}$$

□

PROBLEM: The first line of the above "proof" can easily be read asserting as true exactly what it is trying to prove. Compare that proof with the following.

Theorem 8

$$1 = 0.$$

Incorrect proof. Calculate:

$$\begin{aligned} 1 &= 0 \\ 0 \cdot 1 &= 0 \cdot 0 \\ 0 &= 0. \end{aligned}$$

□

Here is the correct form:

Theorem 9 *Suppose* $x \in \mathbb{R}$. *Then* $(x+1)^2 - (x-1)^2 = 4x$.

Proof. Calculate:

$$\begin{aligned} (x+1)^2 - (x-1)^2 &= (x^2+2x+1) - (x^2-2x+1) \\ &= x^2+2x+1-x^2+2x-1 \\ &= 4x. \end{aligned}$$

□