Review for Midterm *Math 212*

The midterm covers what we have done up through Friday, March 8. It will be in class with closed books/notes. The purpose of the midterm is to give you the chance to review and consolidate the material we have learned so far (as opposed to testing your ingenuity). Please learn the definitions and connections between main ideas carefully. Details appear below.

Definition of the integral. The definitions of the following are the minimum needed to understand the integral of a function over a subset of $\mathbb{R}^n$; they should all be memorized:

- rectangle in $\mathbb{R}^n$
- volume of a rectangle
- partition of a rectangle
- subrectangles of a partition
- upper and lower sums refinement of a partition
- common refinement of two partitions
- bounded function
- lower and upper integral
- integrability of a bounded function over a rectangle in $\mathbb{R}^n$
- integral of a bounded function over a rectangle in $\mathbb{R}^n$
- integral of a bounded function over a bounded subset of $\mathbb{R}^n$
- bounded subset of $\mathbb{R}^n$
- volume of a bounded subset $K \subset \mathbb{R}^n$ (recall: if the integral does not exist, then K does not have volume, which is not the same as volume 0)

Properties of the integral.

- Know how to prove that $L(f, P) \leq U(f, Q)$ for every pair of partitions, P and Q. In other words, every upper sum is larger then every lower sum.
- Know the statement of the integrability criterion (a function is integrable if and only if for all $\epsilon > 0$, etc.).
- Be able to prove that continuous functions are integrable (make sure you know the definition of uniform continuity and the statement of the theorem about continuity on compact sets being automatically uniform).

Fubini and change of variables.

- Review the statement of Fubini, but don't worry about memorizing it. You will need it to evaluate integrals on the test. Recall the order of the integrals in Fubini's theorem can be changed, sometimes resulting in easier integrals.
- Know the statement of the change of variables theorem.
- Memorize the the polar, cylindrical, and spherical changes of coordinates with their stretching factors.

Differential forms. Know how to do all the calculations on the homework due March 8.

Integration practice problems. You can find full solutions at www.reed.edu/~davidp/212.2013.

1. Integrate $x\cos(x+y)$ over the triangle with vertices $(0,0)$, $(\pi,0)$, and (π,π). (Solution: $-3\pi/2$.) For an easier problem, integrate $\cos(x+y)$. (Solution: -2.)
2. Integrate $f(x,y) = x$ over the region bounded by $y = x^2$ and $y = x^3$. (Solution: $1/20$.)
3. Find the volume of the region in $\mathbb{R}^3$ lying above the triangle with vertices $(-1,0,0)$, $(0,1,0)$, and $(1,0,0)$ and under the graph of the function $f(x,y) = x^2 y$. (Solution: $1/30$.)
4. Find the integral of $f(x,y) = (x^2+y^2+1)^{-3/2}$ over the disc of radius a centered at the origin. Show that the limit as $a \to \infty$ is 2π.
5. Find the mass of a sphere of radius a if the density at a point is proportional to the distance from a fixed plane passing through a diameter. (Solution: $k\pi a^4/2$.)
6. What is the area of a triangle with vertices $(1,3)$, $(6,-1)$, and $(7,5)$? (Hint: translate to the origin, then remember that the determinant gives the volume of the box spanned by its rows. Solution: 17.)
7. Let $\phi(u,v) = (u^2 - v^2, 2uv)$. Let S be the region defined by $u^2 + v^2 \leq 1$ and $0 \leq u$, $0 \leq v$. Find the integral of $f(x,y) = (x^2+y^2)^{1/2}$ over $\phi(S)$. (Solution: $\pi/3$.)
8. Let $R = [0,1] \times [0,1]$, and let $\phi(x,y) = (x, x+y^2)$. Sketch $\phi(R)$ and calculate the integral of $f(x,y) = x$ over $\phi(R)$. (Solution: $1/2$.)