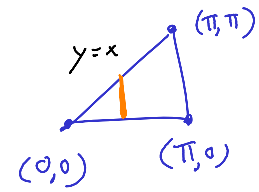
Math 212

Solutions to problems from the midterm review

1.

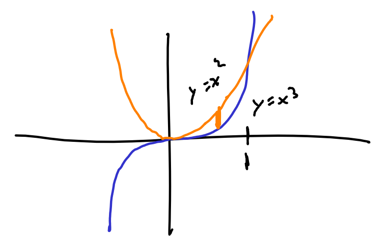


$$\int_0^{\pi}\int_0^{x} x\cos(x+y)\,dy\,dx \;=\; \int_0^{\pi} \left(x\sin(x+y)\right|_{y=0}^{x}\,dx$$
$$=\; \int_0^{\pi} x\sin(2x) - x\sin(x)\,dx.$$

Now integrate by parts, letting $u = x$, $dv = \sin(2x) - \sin(x)$; hence, $du = dx$, $v = -\cos(2x)/2 + \cos(x)$:

$$\int\int \;=\; \left(x(-\cos(2x)/2+\cos(x))\right|_0^{\pi} - \int_0^{\pi} -\cos(2x)/2+\cos(x)\,dx$$
$$=\; -3\pi/2 - \left(-\sin(2x)/4+\sin(x)\right|_0^{\pi}$$
$$=\; -3\pi/2.$$

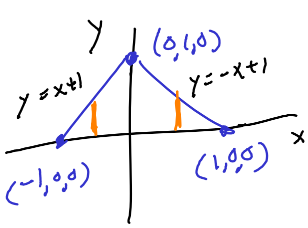
2.



$$\int_0^{1}\int_{x^3}^{x^2} x\,dy\,dx \;=\; \int_0^{1} \left(x\right|_{y=x^3}^{x^2}$$
$$=\; \int_0^1 x^3 - x^4\,dx$$
$$=\; \left(x^4/4 - x^5/5\right|_0^1 = 1/20.$$

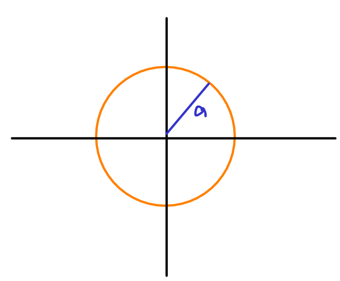
3. vertical slices:

$$\int_{-1}^{0}\int_0^{x+1} x^2y\,dy\,dx \;+\; \int_0^1\int_0^{-x+1} x^2y\,dy\,dx = \int_{-1}^{0}\left(x^2y^2/2\right|_0^{x+1}\,dx + \int_0^1 \left(x^2y^2/2\right|_0^{-x+1}\,dx$$
$$=\; \int_{-1}^{0} x^2(1+x)^2/2\,dx + \int_0^1 x^2(1-x)^2/2\,dx$$
$$=\; = \frac{1}{2}\int_{-1}^{0} x^2+2x^3+x^4\,dx + \frac{1}{2}\int_0^1 x^2-2x^3+x^4\,dx$$
$$=\; \frac{1}{2}\left(x^3/3+x^4/2+x^5/5\right|_{-1}^{0} + \frac{1}{2}\left(x^3/3-x^4/2+x^5/5\right|_0^1$$
$$=\; \frac{1}{2}(1/3-1/2+1/5) + \frac{1}{2}(1/3-1/2+1/5) = 1/30.$$



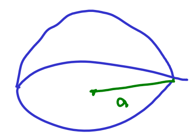
4. Use polar coordinates (stretching factor $= r$).

$$\int_{\text{disc}} (x^2+y^2+1)^{-3/2} \;=\; \int_0^a\int_0^{2\pi} r(r^2+1)^{-3/2}\,d\theta\,dr$$



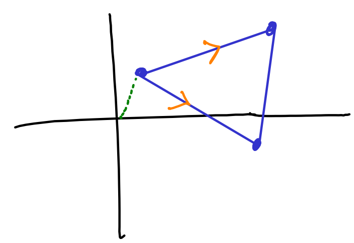
$$
\begin{aligned}
&= 2\pi \int_0^a r(r^2+1)^{-3/2} dr \\
&= 2\pi \left(-(r^2+1)^{-1/2}\Big|_0^a\right. \\
&= 2\pi \left(1 - \frac{1}{\sqrt{a^2+1}}\right) \longrightarrow 2\pi \quad \text{as } a \to 0.
\end{aligned}
$$

5. Let the sphere be centered at the origin, and take the plane through a diameter to be the x, y-plane in x, y, z-space. The mass of a point (x, y, z) is $k|z|$ for some constant k. Use spherical coordinates to integrate $k|z|$ over the sphere of radius a noting that $z = r\cos\phi$ (using the notation for spherical coordinates used in class). We will integrate z over the top half of the sphere and double the result.



$$
\begin{aligned}
2\int_{\text{hemisphere}} z &= \int_0^a \int_0^{\pi/2} \int_0^{2\pi} (kr\cos\phi) r^2 \sin\phi \, d\theta \, d\phi \, dr \\
&= 4k\pi \int_0^a \int_0^{\pi/2} r^3 \cos\phi \sin\phi \, d\phi \, dr \\
&= 4k\pi \int_0^a \left(r^3 \sin^2\phi/2\Big|_0^{\pi/2}\right. dr \\
&= 2k\pi \int_0^a r^3 \, dr \\
&= ka^4\pi/2.
\end{aligned}
$$

6. First translate the vectors to the origin:



$$(6,-1) - (1,3) = (5,-4) \qquad (7,5) - (1,3) = (6,2).$$

The area is the absolute value of

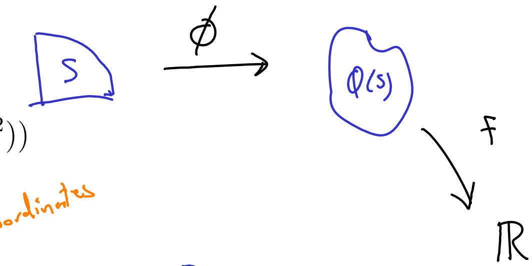
$$\frac{1}{2} \det \begin{bmatrix} 5 & 6 \\ -4 & 2 \end{bmatrix} = -34/2 = -17.$$

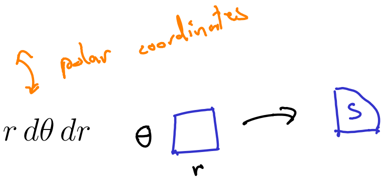
Thus, the area is 17.

7. Use the change of variables formula, then polar coordinates. Note that $\det \phi' = 4(u^2+v^2)$.

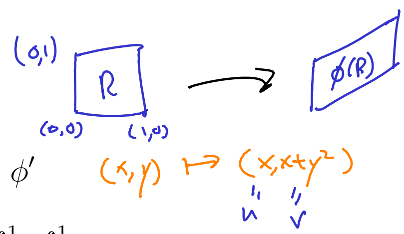
$$
\begin{aligned}
\int_{\phi(S)} f &= \int_S f \circ \phi \, |\det \phi'| \\
&= \int_S ((u^2-v^2)^2 + (2uv)^2)^{1/2} (4(u^2+v^2)) \\
&= \int_S (u^4 + 2u^2v^2 + v^4)^{1/2} (4(u^2+v^2))
\end{aligned}
$$

$$
\begin{aligned}
&= \int_S ((u^2+v^2)^2)^{1/2}(4(u^2+v^2)) \\
&= \int_S 4(u^2+v^2)^2 \\
&= \int_0^1 \int_0^{\pi/2} 4(r^2)^2\, r\, d\theta\, dr \\
&= 2\pi \int_0^1 r^5\, dr \\
&= \pi/3.
\end{aligned}
$$





8. Change of variables:



$$
\begin{aligned}
\int_{\phi(R)} f(x,y) &= \int_R f\circ\phi \det \phi' \\
&= \int_R 2xy = \int_0^1\int_0^1 2xy\, dx\, dy \\
&= 1/2.
\end{aligned}
$$