# Math 111



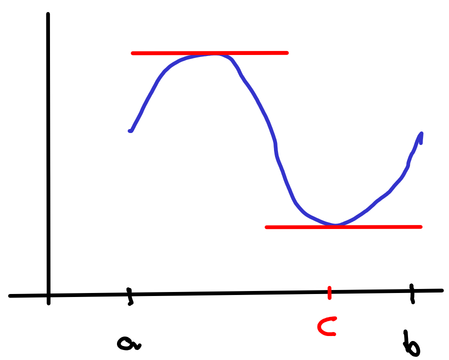
★ Midterm review sheet – online ‖ Today: Proof of the MVT.

**Thm.** (Mean value theorem, MVT) Let $f$ be cts. on $[a,b]$ and diff'ble on $(a,b)$. Then there exists $c \in (a,b)$ such that

$$\text{local} \rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}. \leftarrow \text{global}$$

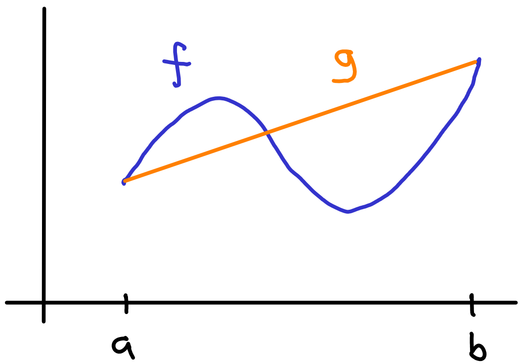
To prove this theorem, we first prove a related theorem.

**Thm.** (Rolle's theorem) If $f$ is cts. on $[a,b]$, diff'ble on $(a,b)$, and $f(a)=f(b)$, then there exists $c \in (a,b)$ such that $f'(c)=0$.




Pf/ By the EVT, $f$ has a min. and max. on $[a,b]$. If they both occur at the endpoints, then since $f(a)=f(b)$, the min. and max are equal. This means $f$ is constant on $[a,b]$, whence $f'=0$ on all of $(a,b)$. So $c$ can be chosen arbitrarily in that case. Otherwise, an extremum occurs in the interior of the interval, where we know the derivative must be $0$. $\square$

**Pf. of the MVT/** Think of $f(x)$ as the position of a bicycle along a road.




Consider another bicycle starting and stopping at the same places as $f$, but travelling at constant speed. The equation for $g$ is:

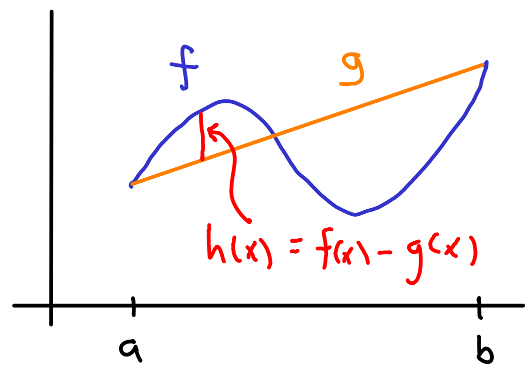
$$g(x) = \left(\frac{f(b)-f(a)}{b-a}\right)(x-a) + f(a).$$

Check: $g(a)=f(a)$, $g(b)=f(b)$, and $g'(x) = \dfrac{f(b)-f(a)}{b-a}$ = average speed of bike $f$.

Intuitively, it's not possible that the speed of bike $g$ is always greater or always less than bike $f$. So there must be a point $c$ where there speeds are the same, i.e. where

$$f'(c) = g'(c) = \frac{f(b)-f(a)}{b-a}.$$

Rolle's theorem makes this reasoning precise. Let $h(x) = f(x) - g(x)$.




Then $h$ is cts. on $[a,b]$, diff'ble on $(a,b)$, and $h(a) = h(b) = 0$. By Rolle's theorem, there exists a point $c \in (a,b)$ where $h'(c) = 0$. But $h'(c) = f'(c) - g'(c)$. So $f'(c) = g'(c) = \dfrac{f(b)-f(a)}{b-a}$. $\square$