Math 212

Intuitive meaning of div, curl, grad.

Div Let $F: \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field, and let $p \in \mathbb{R}^3$. Then div $F(p)$ is the flux density (flux/volume) at p.

Explanation Let B_ε be an ε-ball about p. Then

$$\text{flux of } F \text{ through } \partial B_\varepsilon = \iint_{\partial B_\varepsilon} F\cdot\vec{n} \overset{\text{Stokes'}}{=} \iiint_{B_\varepsilon} \operatorname{div} F$$

$$\approx \operatorname{div} F(p) \cdot \operatorname{vol}(B_\varepsilon)$$

(for ε small so that div F does not vary much on B_ε)

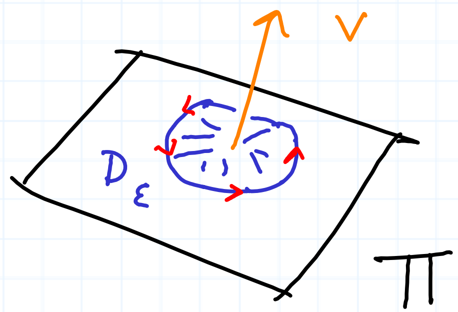
$$\Rightarrow \operatorname{div} F(p) \approx \frac{1}{\operatorname{vol}(B_\varepsilon)} \iint_{\partial B_\varepsilon} F\cdot\vec{n}\,.$$

Take the limit of $\frac{1}{\operatorname{vol}(B_\varepsilon)} \iint_{\partial B_\varepsilon} F\cdot\vec{n}$ to get the flux density at p.

Curl Let $F:\mathbb{R}^3 \to \mathbb{R}^3$ be a vector field, let $p \in \mathbb{R}^3$, and let $v \in \mathbb{R}^3$ be a unit vector (i.e., a direction). Then

$\operatorname{curl} F(p) \cdot v$ = circulation density (circulation/area) of F about the point p in the plane through p perpendicular to v.

Explanation Let Π be the plane perpendicular to v and containing p. Let D_ε be the ε-disc centered at p and lying in Π. Then,



$$\text{circulation of } F \text{ about } \partial D_\varepsilon = \int_{\partial D_\varepsilon} F \cdot \vec{t}$$

$$\overset{\text{Stokes'}}{=} \iint_{D_\varepsilon} \operatorname{curl} F \cdot \vec{n} \quad \leftarrow \vec{n} = v$$

$$\approx \operatorname{area}(D_\varepsilon)\, \operatorname{curl} F(p) \cdot v$$

(for ε small so that $\operatorname{curl} F(p)$ does not vary much).

$$\Longrightarrow \quad \operatorname{curl} F(p) \cdot v \simeq \frac{1}{\text{area}(D_\varepsilon)} \int_{\partial D_\varepsilon} F \cdot t \,.$$

Take the limit of $\frac{1}{\text{area}(D_\varepsilon)} \int_{\partial D_\varepsilon} F \cdot t$ as $\varepsilon \to 0$ to get circulation density.

Grad Let $\varphi : \mathbb{R}^3 \to \mathbb{R}$. Then φ is the potential for $\operatorname{grad} \varphi : \mathbb{R}^3 \to \mathbb{R}^3$.
Let $v \in \mathbb{R}^3$ be a unit vector (i.e., a direction). Then

$\operatorname{grad} \varphi(p) \cdot v$ is the "change density" (change/length) of φ at p in the direction of v.

Explanation From Math 211, you know $\operatorname{grad} \varphi(p) \cdot v$ is the directional derivative:

$$\lim_{\varepsilon \to 0} \frac{\varphi(p + \varepsilon v) - \varphi(p)}{\varepsilon},$$

which captures the idea of "change density" at p in the direction of v.

One may also give an argument using Stokes', as previously. Let $C(t) = p + tv$ for $t \in [0, \varepsilon]$. Then the change in potential is

$$\underbrace{Q(C(\varepsilon))}_{= Q(p+\varepsilon v)} - \underbrace{Q(C(0))}_{= Q(p)} \overset{\text{Stokes'}}{=} \int_C \operatorname{grad} Q \cdot \underbrace{\vec{n}}_{= v} \approx \operatorname{grad} Q(p) \cdot v \; \overbrace{\operatorname{length}(C)}^{= \varepsilon}$$

$$\Longrightarrow \operatorname{grad} Q(p) \cdot v \approx \frac{Q(p+\varepsilon v) - Q(p)}{\varepsilon}.$$

Take the limit of $\dfrac{Q(p+\varepsilon v) - Q(p)}{\varepsilon}$ as $\varepsilon \to 0$ to get the "change density", i.e. the directional derivative of Q at p in the direction of v.