Math 212 Integrals

Weighted n-dimensional volume of a subset of $\mathbb{R}^n$

$D \subseteq \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$ (weighting function)

$\boxed{\mathrm{vol}_f(D) := \int_D f}$ ← Defined via upper and lower sums of partitions

$\int_D f \approx \sum_J f(x_J)\,\mathrm{vol}(J)$. ← Riemann sum

$x_J \in J$

J: subbox of a partition

The integral can be calculated exactly using Fubini's thm. (multiple integrals).

Weighted surface area (k-dimensional volume) of a k-surface

$D \subseteq \mathbb{R}^k$, $\underbrace{S : D \to \mathbb{R}^n}_{k\text{-surface}}$, $\underbrace{f : \mathbb{R}^n \to \mathbb{R}}_{\text{weighting function}}$

$\boxed{\mathrm{area}_f(S) := \int_D f \circ S \sqrt{\det g}}$ where g is the $k \times k$ matrix with ij^{th} entry $\frac{\partial S}{\partial x_i} \cdot \frac{\partial S}{\partial x_j}$.

Special Cases:

1) weighted length of a curve in $\mathbb{R}^n$

$$C: [a,b] \to \mathbb{R}^n, \qquad f: \mathbb{R}^n \to \mathbb{R}$$

$$\boxed{\text{length}_f(C) = \text{area}(C) = \int_C f := \int_D f \circ C \, |C'|}.$$

2) weighted area of a surface in $\mathbb{R}^3$

$$D \subseteq \mathbb{R}^2, \quad S: D \to \mathbb{R}^3, \quad f: \mathbb{R}^3 \to \mathbb{R}$$

$$S_u = \frac{\partial S}{\partial u}, \quad S_v = \frac{\partial S}{\partial v}.$$

$$\boxed{\text{area}_f(S) = \int_S f := \int_D f \circ S \, |S_u \times S_v|}.$$

3) weighted n-dimensional volume of an n-surface

$$D \subseteq \mathbb{R}^n, \quad V: D \to \mathbb{R}^n,$$

$$\boxed{\text{area}_f(V) = \text{vol}_f(V) = \int_V f := \int_D f \circ V \, |\det JV|}$$

If V is 1-1 except on a set of volume zero, then

$$\int_V f = \int_{V(D)} f .$$

weighted n-dimensional volume of the set $V(D) \subseteq \mathbb{R}^n$

This comes from the change of variables theorem:

$$D \xrightarrow{V} V(D) \xrightarrow{f} \mathbb{R}$$

$$\boxed{\int_{V(D)} f = \int_D f \circ V \, |\det JV|} .$$

Stokes' theorem, vector fields, gradient, curl, divergence

$\omega \in \Omega^k \mathbb{R}^n$ k-form in $\mathbb{R}^n$

$D \subseteq \mathbb{R}^k$, $S : D \to \mathbb{R}^n$ k-surface in $\mathbb{R}^n$

$$\boxed{\int_S w := \int_D S^* w}$$

$\leftarrow$ Substitute the i^{th} component of S, $S_i(u_1, \dots, u_k)$ for the i^{th} variable, x_i. Write $S^* w = g \, du_1 \wedge \cdots \wedge du_k$ for some g. Then $\int_D S^* w := \int_D g =$ weighted k-dimensional volume of D.

Stokes' If $w \in \Omega^{k-1} \mathbb{R}^n$, a $(k-1)$-form, and $S : D \to \mathbb{R}^n$ is a k-surface, then

$$\boxed{\int_{\partial S} w = \int_S dw},$$

where ∂S is the boundary of S.

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Flow of a vector field along a curve (work if F is thought of as a force)

$C : [a,b] \to \mathbb{R}^n$ curve in $\mathbb{R}^n$

$F : \mathbb{R}^n \to \mathbb{R}^n$ vector field in $\mathbb{R}^n$

component of F along C

$$= \int_a^b \left[(F \circ C) \cdot \frac{C'}{|C'|} \right] |C'| = \text{weighted length of } C$$

$\omega_F = F_1\, dx_1 + \cdots + F_n\, dx_n$ flow form

dot product

$$\boxed{\text{flow of } F \text{ along } C = \int_C F \cdot \vec{t} := \int_C \omega_F = \int_a^b (F \circ C) \cdot C'.}$$

Flux of a vector field through a hypersurface

$D \subseteq \mathbb{R}^{n-1}$, $S : D \to \mathbb{R}^n$ hypersurface in $\mathbb{R}^n$

$F : \mathbb{R}^n \to \mathbb{R}^n$ vector field in $\mathbb{R}^n$

$$\omega^F = \sum_{i=1}^n (-1)^{i+1} F_i\, dx_1 \wedge \cdots \wedge \overline{dx_i} \wedge \cdots \wedge dx_n$$

omit

dot product

$$\boxed{\text{Flux of } F \text{ through } S = \int_S F \cdot \vec{n} := \int_S \omega^F = \int_D (F \circ C) \cdot (S_{u_1} \times \cdots \times S_{u_{n-1}})}$$

where $S_{u_1} \times \cdots \times S_{u_n} = \det \begin{bmatrix} e_1 & e_2 & \cdots & e_n \\ S_{u_1 1} & S_{u_1 2} & \cdots & S_{u_1 n} \\ & \vdots & & \\ S_{u_{n-1} 1} & S_{u_{n-1} 2} & \cdots & S_{u_{n-1} n} \end{bmatrix} \begin{array}{l} \\ \leftarrow S_{u_1} \\ \vdots \\ \leftarrow S_{u_{n-1}} \end{array}$

Classical case: $n=3$

$$\omega^F = F_1\, dy \wedge dz - F_2\, dx \wedge dz + F_3\, dx \wedge dy$$

$$\underset{\uparrow \atop u_1}{S_u} \times \underset{\uparrow \atop u_2}{S_v} = \det \begin{bmatrix} i & j & k \\ S_{u1} & S_{u2} & S_{u3} \\ S_{v1} & S_{v2} & S_{v3} \end{bmatrix}$$

$$\boxed{\int_S F \cdot \vec{n} = \int_D (F \circ S) \cdot (S_u \times S_v)}$$

$$\| $$

$$\int_D \underbrace{\Big((F \circ S) \cdot \underset{\uparrow \atop \text{unit normal to } S}{\frac{S_u \times S_v}{|S_u \times S_v|}} \Big)}_{\text{component of } F \text{ normal to } S} |S_u \times S_v|$$

$$\| $$

surface area of S weighted by component of F normal to S.

Classical Cases of Stokes' Theorem

$\cancel{n=0,1}$

$\varphi : \mathbb{R}^n \to \mathbb{R}$ 0-form, potential function

$\nabla \varphi = \operatorname{grad} \varphi = \left(\frac{\partial \varphi}{\partial x_1}, \ldots, \frac{\partial \varphi}{\partial x_n} \right)$ gradient vector field $\mathbb{R}^n \to \mathbb{R}^n$

$d\varphi = \sum \frac{\partial \varphi}{\partial x_i} dx_i = \omega_{\nabla \varphi}$ = flow form for $\nabla \varphi$.

Stokes: $C : [a,b] \to \mathbb{R}^n$ curve in $\mathbb{R}^n$

$$\boxed{\int_C \nabla \varphi \cdot \vec{t} = \int_C d\varphi = \int_{\partial C} \varphi = \varphi(C(b)) - \varphi(C(a))}$$

The flow of a gradient vector field along a curve C is equal to the change in potential.

$n = 1, 2$

$\nabla = (D_1, D_2, D_3)$

$F : \mathbb{R}^3 \to \mathbb{R}^3$

$D_1 = \frac{\partial}{\partial x}, \; D_2 = \frac{\partial}{\partial y}, \; D_3 = \frac{\partial}{\partial z}$

$$\nabla \times F = \operatorname{curl} F = \det \begin{bmatrix} i & j & k \\ D_1 & D_2 & D_3 \\ F_1 & F_2 & F_3 \end{bmatrix} = (D_2F_3 - D_3F_2,\ D_3F_1 - D_1F_3,\ D_1F_2 - D_2F_1)$$

$$d\underbrace{\tilde{\omega}_F}_{\text{flow form for } F} = \underbrace{\omega^{\nabla \times F}}_{\text{flux form for } \nabla \times F}$$

Stokes' $D \subseteq \mathbb{R}^2$, $S : D \to \mathbb{R}^3$ surface in $\mathbb{R}^3$

$$\boxed{\int_S (\nabla \times F) \cdot \vec{n} = \int_S \omega^{\nabla \times F} = \int_S d\omega_F = \int_{\partial S} \omega_F = \int_{\partial S} F \cdot \vec{t}}$$

The flux of the curl F through S equals the circulation of F along the boundary of S.

$n = 2, 3$

$$F : \mathbb{R}^3 \to \mathbb{R}^3$$

$$\nabla \cdot F = \operatorname{div} F = D_1 F_1 + D_2 F_2 + D_3 F_3 \qquad \operatorname{div} F : \mathbb{R}^3 \to \mathbb{R}$$

$$d\omega^F = \nabla \cdot F \; dx \wedge dy \wedge dz$$

flux form for F

Stokes' $\quad D \subseteq \mathbb{R}^3, \quad V : D \to \mathbb{R}^3$ with $\det JV \geq 0$.

$$\boxed{\int_V \nabla \cdot F = \int_V d\omega^F = \int_{\partial V} \omega^F = \int_{\partial V} F \cdot \vec{n}}$$

The volume of V weighted by the div F equals the flux of F through the boundary of V.

Geometric Meaning of grad, curl, div

gradient $v \in \mathbb{R}^3$ unit vector, $p \in \mathbb{R}^3$

change in φ

$=$ change density

rate of change

* $(\nabla \varphi)(p) \cdot v$ = directional derivative of φ in the direction of v

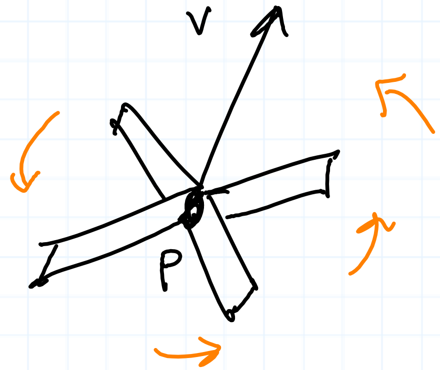
$$= \lim_{t \to 0} \frac{\varphi(p+tv) - \varphi(p)}{t}$$

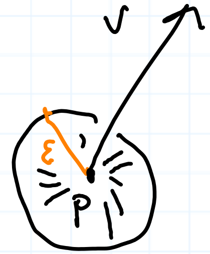
* $\nabla \varphi(p)$ points in the direction of quickest increase of φ at p
* $\nabla \varphi$ is perpendicular to the level set through p, $\varphi^{-1}(p)$

curl $v \in \mathbb{R}^3$ unit vector, $p \in \mathbb{R}^3$

* $(\nabla \times F) \cdot v$ = circulation density of F around v in the plane perpendicular to v passing through p.

* $\nabla \times F$ points in the direction about which F spins the fastest.





$$(\nabla \times F)\cdot v = \lim_{\varepsilon\to 0} \frac{1}{\operatorname{area}(D_\varepsilon)} \int_{\partial D_\varepsilon} F\cdot\vec{t}\,.$$

D_ε = disk of radius ε perpendicular to p

divergence $\quad p \in \mathbb{R}^3$

$\nabla\cdot F(p)$ = flux density at p.

$$\operatorname{div} F(p) = \lim_{\varepsilon\to 0} \frac{1}{\operatorname{vol}(B_\varepsilon)} \int_{\partial B_\varepsilon} F\cdot\vec{n}$$

B_ε = ball of radius ε