Theorem. *Let f be an integrable function on $[a, b]$. Suppose there is a function g, continuous on $[a, b]$ and differentiable on (a, b), such that $g' = f$ on (a, b). Then,*

$$\int_a^b f = g(b) - g(a).$$

In other words, $\int_a^b g' = g(b) - g(a)$.

Proof. Let $P = \{t_0, \dots, t_n\}$ be any partition of $[a, b]$. Apply the mean value theorem to g on each subinterval of P. For each $i = 1, \dots, n$, we get $c_i \in [t_{i-1}, t_i]$ such that

$$g'(c_i) = \frac{g(t_i) - g(t_{i-1})}{t_i - t_{i-1}}.$$

Since $g'(c_i) = f(c_i)$, we can substitute and rearrange to get

$$f(c_i)(t_i - t_{i-1}) = g(t_i) - g(t_{i-1}), \tag{1}$$

for each i.

Let $M_i = \sup\{f(x) : x \in [t_{i-1}, t_i]\}$ and $m_i = \inf\{f(x) : x \in [t_{i-1}, t_i]\}$, as usual. It follows that

$$m_i \le f(c_i) \le M_i \quad \Rightarrow \quad m_i(t_i - t_{i-1}) \le f(c_i)(t_i - t_{i-1}) \le M_i(t_i - t_{i-1}),$$

for each i. Summing over i,

$$L(f, P) \le \sum_{i=1}^n f(c_i)(t_i - t_{i-1}) \le U(f, P).$$

Using equation (1),

$$L(f, P) \le \sum_{i=1}^n (g(t_i) - g(t_{i-1})) \le U(f, P).$$

But $\sum_{i=1}^n (g(t_i) - g(t_{i-1}))$ is a telescoping sum with value $g(b) - g(a)$. Thus,

$$L(f, P) \le g(b) - g(a) \le U(f, P)$$

for each partition P. It follows that

$$\sup_P\{L(f, P)\} \le g(b) - g(a) \le \inf_P\{U(f, P)\}$$

However, since f is integrable,

$$\sup_P\{L(f, P)\} = \inf_P\{U(f, P)\} = \int_a^b f.$$

The result follows. □