The *determinant* of an $n \times n$ matrix M is the unique multilinear, alternating function, det, of the rows of the matrix such that $\det I_n = 1$ (where I_n is the $n \times n$ identity matrix). All of the properties of the determinant follow from those properties. *Multilinear* means that the function is linear with respect to each argument. Thus, if $r_1, \ldots, r_n$ are the row vectors (elements of $\mathbb{R}^n$), r_i' is another row vector, and $s \in \mathbb{R}$, we have

$$\begin{aligned}\det(r_1, \ldots, r_{i-1}, s\, r_i + r_i', r_{i+1}, \ldots, r_n) &= s\ \det(r_1, \ldots, r_{i-1}, r_i, r_{i+1}, \ldots, r_n)\\ &\quad + \det(r_1, \ldots, r_{i-1}, r_i', r_{i+1}, \ldots, r_n).\end{aligned}$$

Alternating means that the determinant is zero if two of its arguments are equal:

$$\det(r_1, \ldots, r_n) = 0$$

if $r_i = r_j$ for some $i \neq j$. A consequence is that swapping two arguments causes the determinant to change just by a sign:

$$\det(r_1, \ldots, r_i, \ldots, r_j, \ldots, r_n) = -\det(r_1, \ldots, r_j, \ldots, r_i, \ldots, r_n).$$

(Hint for a proof: in $\det(r_1, \ldots, r_i, \ldots, r_j, \ldots, r_n)$ replace both r_i and r_j by $r_i + r_j$. This determinant must be 0. Now expand using multilinearity.) Another consequence is that the determinant does not change if a multiple of one row is added to another. For ease of typesetting, we will show this for the first two rows:

$$\begin{aligned}\det(r_1 + s\, r_2, r_2, \ldots, r_n) &= \det(r_1, r_2, \ldots, r_n) + s\ \det(r_2, r_2, r_3, \ldots, r_n)\\ &= \det(r_1, r_2, \ldots, r_n) + 0 = \det(r_1, \ldots, r_n).\end{aligned}$$

Example. Here we compute the determinant of a 2×2 matrix using the fact that the determinant is a multilinear alternating mapping with value 1 on the identity matrix.

$$\begin{aligned}\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \det((a, b), (c, d))\\ &= \det(a\, e_1 + b\, e_2, c\, e_1 + d\, e_2)\\ &= a\ \det(e_1, c\, e_1 + d\, e_2) + b\ \det(e_2, c\, e_1 + d\, e_2)\\ &= ac\ \det(e_1, e_1) + ad\ \det(e_1, e_2) + bc\ \det(e_2, e_1) + bd\ \det(e_2, e_2)\\ &= 0 + ad\ \det(e_1, e_2) + bc\ \det(e_2, e_1) + 0\\ &= ad\ \det(e_1, e_2) - bc\ \det(e_1, e_2)\\ &= ad\ \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - bc\ \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\\ &= ad \cdot 1 - bc \cdot 1 = ad - bc.\end{aligned}$$

Facts.

1. Define the *transpose*, M^t of M by $M^t_{ij} := M_{ji}$. Then $\det M^t = \det M$.
2. The determinant is the unique multilinear alternating function on the *columns* of a matrix.
3. The determinant may be calculated by "expanding" along any row or column.
4. $\det M = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) M_{1\sigma(1)} \cdots M_{n\sigma(n)}$, where S_n is the collection of all permutations of $(1, \ldots, n)$ and σ gives the *sign* of the permutation (i.e., 1 if the permutation is formed by an even number of flips and -1 if it is formed by an odd number of flips).
5. The determinant gives the signed volume of the parallelepiped spanned by the rows (or by the columns) of the matrix.