HW 11 for Math 212

Due Friday, May 3

1. Verify Stokes' theorem, $\iint_S \operatorname{curl} \mathbf{F} \cdot \vec{\mathbf{n}} = \int_{\partial S} \mathbf{F} \cdot \vec{\mathbf{t}}$, for the vector field $\mathbf{F}(x, y, z) = (y, z, x)$ (*note the weird order of the variables*) and the surface S given by the octant of the sphere $x^2 + y^2 + z^2 = 1$ where x, y, and z are all nonnegative. The boundary consists, essentially, of three arcs of circles, one in each of the coordinate planes. (I believe you will need to parametrize the surface, i.e., there is no short-cut based on the geometry.)

2. For a parametrized surface $S(u, v)$ with $(u, v) \in D \subset \mathbb{R}^2$, we defined the surface area
$$\operatorname{area}(S) := \int_D |S_u \times S_v|.$$
Show that the surface area of the graph $z = f(x, y)$ for $(x, y) \in D \subset \mathbb{R}^2$ is
$$\int_D \sqrt{1 + f_x^2 + f_y^2},$$
where f_x and f_y are the partials of f.

3. Let B_s be a box of side-length s centered at the point $p = (p_1, p_2, p_3) \in \mathbb{R}^3$, and consider the vector field, $\mathbf{F} = x^2\vec{\mathbf{i}} + y^2\vec{\mathbf{j}} + z^2\vec{\mathbf{k}}$. Show that
$$\lim_{s \to 0} \frac{1}{\operatorname{vol} B_s} \iint_{\partial B_s} \mathbf{F} \cdot \vec{\mathbf{n}} = \operatorname{div} \mathbf{F}(p)$$
by directly calculating the flux (do not use Stokes' theorem).

4. (a) Using Maxwell's equations, show that we can write
$$\mathbb{B} = \nabla \times \mathbb{A},$$
$$\mathbb{E} = -\nabla \Phi - \frac{\partial \mathbb{A}}{\partial t},$$
where $\mathbb{A}$ (the *vector potential*) is some vector function of position and time, and Φ (the *scalar potential*) is some scalar function of position and time. (**Note:** you may assume that $\mathbb{B}$ and $\mathbb{E}$ are smooth fields defined on all of $\mathbb{R}^3$.)

(b) Show that $\mathbb{A}$ and Φ satisfy

$$\nabla^2\Phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbb{A}) = -\rho/\epsilon_0,$$
$$\nabla^2\mathbb{A} - \mu_0\epsilon_0\frac{\partial^2\mathbb{A}}{\partial^2 t} = -\mu_0\mathbb{J} + \nabla\left(\nabla \cdot \mathbb{A} + \mu_0\epsilon_0\frac{\partial\Phi}{\partial t}\right).$$

(c) Show that if we define two new potentials

$$\mathbb{A}' = \mathbb{A} + \nabla\chi,$$
$$\Phi' = \Phi - \frac{\partial\chi}{\partial t},$$

where χ is an arbitrary scalar function of position and time, then

$$\mathbb{B} = \nabla \times \mathbb{A}',$$
$$\mathbb{E} = -\nabla\Phi' - \frac{\partial\mathbb{A}'}{\partial t}.$$

Thus, $\mathbb{B}$ and $\mathbb{E}$ are not modified by the change in potentials. The change from $(\mathbb{A}, \Phi)$ to $(\mathbb{A}', \Phi')$ is called a *gauge transformation*.

(d) Show that if χ is required to satisfy

$$\nabla^2\chi - \mu_0\epsilon_0\frac{\partial^2\chi}{\partial^2 t} = -\left(\nabla \cdot \mathbb{A} + \mu_0\epsilon_0\frac{\partial\Phi}{\partial t}\right),$$

then

$$\nabla \cdot \mathbb{A}' + \mu_0\epsilon_0\frac{\partial\Phi'}{\partial t} = 0.$$

(e) If χ satisfies the equation in (4d), show that $\mathbb{A}'$ and Φ' satisfy the equations

$$\nabla^2\mathbb{A}' - \mu_0\epsilon_0\frac{\partial^2\mathbb{A}'}{\partial^2 t} = -\mu_0\mathbb{J},$$
$$\nabla^2\Phi' - \mu_0\epsilon_0\frac{\partial^2\Phi'}{\partial^2 t} = -\rho/\epsilon_0.$$

The point of this problem is that after applying an appropriate gauge transformation, one may assume the vector and scalar potentials satisfy wave equations with source terms proportional to the current density and charge density, respectively.

5. The heat Q in a body of volume V is given by

$$Q = c \iiint_V T \rho \, dV,$$

where c is a constant called the *specific heat* of the body, and $T(x, y, z, t)$ and $\rho(x, y, z)$ are, respectively, the temperature and mass density of the body (we are assuming the density is independent of time). The rate at which heat flows through the boundary $S := \partial V$ is

$$\frac{dQ}{dt} := k \iint_S \nabla T \cdot \vec{\mathbf{n}},$$

where k is the *thermal conductivity* of the body, assumed constant. Derive the heat flow equation:

$$\nabla^2 T = \alpha \frac{\partial T}{\partial t},$$

where $\alpha = c\rho/k$.