HW 10 for Math 212
Due Friday, April 26

1. Find the surface area of $S(u,v) = (u, v, u^2+2v, u^2-2v)$ for $u, v \in [0,1]$. (Set up the integral.)

2. Let $F = (F_1, F_2, F_3)$ be a vector field in $\mathbb{R}^3$, and let $\phi\colon \mathbb{R}^3 \to \mathbb{R}$.

   (a) Let $\phi\colon \mathbb{R}^3 \to \mathbb{R}$. Show that $\text{curl}(\text{grad}(\phi)) = 0$ in two ways: (i) by direct calculation using the classical definitions, and (ii) by making the connection with differential forms explicit and using the fact that $d^2 = 0$.

   (b) For a vector field $F = (F_1, F_2, F_3)$ on $\mathbb{R}^3$ show $\text{div}(\text{curl}(F)) = 0$ in two ways as in the previous problem.

3. For a differential $k$-form $\omega$ defined on a "simply connected" (= no holes) open subset of $\mathbb{R}^n$, it turns out the $d\omega = 0$ if and only if there exists a $(k-1)$-form $\lambda$ such that $\omega = d\lambda$. By carefully making the connection with differential forms explicit, show that for a vector field $F$ defined on a simply connected open subset of $\mathbb{R}^3$

   (a) $\text{curl}\, F = 0$ implies $F$ has a potential;

   (b) $\text{div}\, F = 0$ implies $F$ has a vector potential $G$ (meaning $F = \text{curl}\, G$).

4. (a) Let $F = (F_1, F_2, F_3)$ be a vector field on $\mathbb{R}^3$. Prove that

$$\nabla \times (\nabla \times F) = \nabla(\nabla \cdot F) - \nabla^2 F.$$

   Note that $\nabla^2 := \nabla \cdot \nabla = (D_1, D_2, D_3) \cdot (D_1, D_2, D_3) = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$. Hence,

$$\nabla^2 F = \left( \frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}, \frac{\partial^2 F_2}{\partial x^2} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_2}{\partial z^2}, \frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_3}{\partial z^2} \right),$$

   the Laplacian of $F$.

   (b) Suppose that $\mathbb{J} = 0$ and $\rho = 0$ in Maxwell's equations. Use Maxwell's equations and the identity you just established to show that the magnetic field, $\mathbb{B}$, satisfies the wave equation in three dimensions:

$$\nabla^2 \mathbb{B} = \frac{1}{v^2} \frac{\partial^2 \mathbb{B}}{\partial t^2},$$

   for some real number $v$.

5. (a) Show that the divergence of a gradient vector field on $\mathbb{R}^3$ is given by the Laplacian of the potential.

   (b) Show that Maxwell's equations imply that in the absence of a time-varying magnetic field, that (i) the electric field has a potential, and (ii) the Laplacian of the potential is proportional to the charge density. [Note: the *electrostatic potential* is by definition the negative of the potential in part (i).]

6. Let $\mathbb{B}(x, y, z) = (x^2, -y, z)$. Why can't $\mathbb{B}$ be a magnetic field?

7. Here is a "proof" that magnetism does not exist. One of Maxwell's equations says that $\nabla \cdot \mathbb{B} = 0$. On $\mathbb{R}^n$ we always have $d\omega = 0$ for a $k$-form $\omega$ if and only if $\omega = d\lambda$ for some $(k-1)$-form $\lambda$. Hence, there is a vector field $\mathbb{A}$ such that $\mathbb{B} = \nabla \times \mathbb{A}$. From the divergence theorem, we get $\int\int_S \mathbb{B} = \int\int\int_V \nabla \cdot \mathbb{B} = 0$ for any solid $V$ with boundary $S$. By the classical Stokes' theorem, it follows that $0 = \int\int_S \mathbb{B} = \int\int_S \nabla \times \mathbb{A} = \int_C \mathbb{A}$, where $C$ is the boundary of $S$. Hence, $\mathbb{A}$ is conservative, i.e., its flow along all closed loops is zero. Therefore, $\mathbb{A}$ has a potential function: $\mathbb{A} = \nabla\phi$. It follows that

$$\mathbb{B} = \nabla \times \mathbb{A} = \nabla \times (\nabla\phi) = 0.$$

   Therefore, all magnetic fields are zero. What's wrong?

8. An electric field is given by $\mathbb{E}(x, y, z) = (x, y, 0)$. Use Maxwell's equations to determine the amount of charge in the half-cylinder of length $2s$ and radius $r$ shown below: