HW 9, due Friday, April 19 *Math 212*

1. Let $C(t) = (e^{-t}\cos t, e^{-t}\sin t)$.

   (a) Sketch the image of C for some reasonable range for t.

   (b) Let $a > 0$, and calculate the length of C for $t \in [0, a]$.

   (c) Take the limit of your previous calculation as $a \to \infty$ to find the length of the curve for $t \in [0, \infty)$.

2. Fix $a > 0$. The function $\Phi\colon [0,1]^2 \to \mathbb{R}^3$ given by

$$\Phi(u, v) = (a(1 - u - v), 2au, av)$$

   parametrizes a parallelogram in $\mathbb{R}^3$. Let $F(x, y, z) = (0, 0, y)$ be a vector field on $\mathbb{R}^3$.

   (a) Sketch the boundary of Φ. (It helps to first identify the image of the corners of the square.)

   (b) Verify Stokes' theorem by showing that the circulation about the perimeter of the parallelogram is equal to the flux of the curl of F through the parallelogram. (Two calculations.)

   (c) Take the value you get in part (b), divide by the area of the parallelogram, and take the limit as $a \to 0$. (Recall that if v and w are vectors (at the origin) in $\mathbb{R}^3$, then the area of the parallelogram determined by v and w is $|v \times w|$.)

   (d) Show that the answer to part (c) is equal to $\hat{n} \cdot \operatorname{curl} F$, where

$$\hat{n} = \frac{D_1\Phi \times D_2\Phi}{|D_1\Phi \times D_2\Phi|}$$

   is the unit normal to the parallelogram pointing away from the origin. The point being illustrated here is that the curl measures "circulation density."

3. Let $f(x, y, z) = x^2 + 2y + 3z$. Integrate f over the following:

   (a) The set $[0, 1]^3$;

   (b) The curve $C(t) = (t, 2t, 3t)$, $t \in [0, 1]$;

   (c) The surface $S(u, v) = (u + v, u - v, v)$, $u, v \in [0, 1]$;

   (d) The solid $V(x, y, z) = (x + 2y + z, x - y, y + z)$, $x, y, z \in [0, 1]$.

4. Let $F(x, y, z) = (2x, 2yz + x^2, y^2)$.

   (a) Does F have a potential function? (Hint: $d\omega_F = ?$)

(b) Is F the curl of another vector field? (Hint: $d\omega^F = ?$)

5. Let V be a blob in $\mathbb{R}^3$ with volume v. Let $F(x, y, z) = (x, 2y, 3z)$. What is the flux of F through the boundary of V?

6. Let C be a cylinder in $\mathbb{R}^3$ defined by $x^2+y^2 \leq 4$ and $0 \leq z \leq 2$. Let $F(x, y, z) = (0, 0, z)$.

   (a) Without parametrizing C, directly—without the use of Stokes' theorem—calculate the flux of F through the boundary of C just from geometric considerations. Explain your reasoning.

   (b) Use the divergence of F to check your answer.

7. Let $G(x, y, z) = (x \cos y, y \sin x, yz)$, and let $F = \nabla \times G = \operatorname{curl} G$. Find the flux of F through the sphere of radius 5 centered at the origin.

8. Suppose you are sitting in a room in the shape of a rectangular box, and you have set up a coordinate system with yourself as the origin. Let $F(x, y, z) = (x, y, z)$. Show that you can use the flux of F through the walls (and floor) of the room to calculate its volume.

9. Let $F(x, y, z) = (x^2, x + y^2, yz + z^3)$, and let $p = (3, 2, 1)$.

   (a) What is the flux per unit volume of F at the point p?

   (b) At the point p, in which direction would you point the axis of a tiny paddle wheel, pushed by F, to get the wheel spinning the fastest?