HW 8, due Friday, April 12 *Math 212*

1. Let $D = [0,1]^2$, and let $S \colon D \to \mathbb{R}^2$ be a parametrization of a solid region in the plane . Let $F(x,y) = (F_1(x,y), F_2(x,y))$ be a vector field in the plane.

   (a) Prove Green's theorem:

   $$\oint_{\partial S} F_1\,dx + F_2\,dy = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\,dx \wedge dy.$$

   (b) Name a simple condition on F so that its circulation about the boundary will give the area enclosed by S.

   (c) Give an example of a vector field F having this property.

   (d) As a simple test case, calculate the area of a triangle with vertices $(0,0)$, $(a,0)$, and $(0,b)$ by calculating the flow of your F along edges of the triangle.

2. One may parametrize the surface of a sphere of radius r by

   $$S(\theta,\psi) = (r\cos(\theta)\sin(\psi), r\sin(\theta)\sin(\psi), r\cos(\psi))$$

   with $0 \le \theta \le 2\pi$ and $0 \le \psi \le \pi$.

   (a) Calculate $|S_\theta \times S_\psi|$.

   (b) Calculate the surface area of the sphere.

3. Let $S(u,v) = (u, v, u^2 - v^2)$ for $(u,v) \in D = [0,1]^2$, let $C(t) = (t, t^2, t^3)$ for $t \in I = [0,1]$, and let $F(x,y,z) = (y^2 + z, x^2 + y^2, x)$.

   (a) Calculate the flux of F through S in two ways: (i) converting F to a 2-form and integrating that form over S, and (ii) using the formula $\int_S F \cdot \vec{n} = \int_D (F \circ S) \cdot (S_u \times S_v)$.

   (b) Calculate the flow of F around C in two ways: (i) converting F to a 1-form and integrating that form over C, and (ii) using the formula $\int_C F \cdot \vec{t} = \int_I (F \circ C) \cdot C'$.

4. If $\omega = d\lambda$ for some form λ, then $d\omega = 0$. This problem shows that the converse does not hold. Let

   $$\omega = \frac{x\,dy - y\,dx}{x^2 + y^2} = \frac{x}{x^2 + y^2}\,dy - \frac{y}{x^2 + y^2}\,dx.$$

   (a) Show that $d\omega = 0$.

   (b) Show that $\omega \ne d\lambda$ for any function (0-form) λ. (Hint: show that $\int_\gamma \omega \ne 0$ for the closed curve $\gamma(t) = (\cos(t), \sin(t))$ with $0 \le t \le 2\pi$. Stokes' theorem implies that if $\omega = d\lambda$, then $\int_\gamma \omega = 0$.)

5. Let $D = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\}$, the closed unit disk in the plane. Consider the surface $S\colon D \to \mathbb{R}^3$ with $S(u, v) = (u, v, -u^2 - v^2 + 1)$ and the vector field $F(x, y, z) = (-y, x, z)$.

   (a) Calculate $S_u \times S_v$.

   (b) Sketch the image of S and draw $S_u \times S_v$ at some point. (Make sure you get the direction—the sign—correct.)

   (c) Calculate the flux of the curl of F through S.

   (d) Parametrize the boundary of the image of S. (Do not worry about the formal definition of the boundary. You will want to get the direction right so that the right-hand rule will point in the direction of the normal vector to S you already calculated.)

   (e) Calculate the circulation (i.e., flow) of F along this boundary.