HW 7, due Friday, April 5 *Math 212*

1. What is the volume of the cored apple:
$$V := \{(x, y, z) \mid x^2 + y^2 + z^2 \le R^2,\ x^2 + y^2 \ge r^2\},$$
where $0 < r < R$?

2. (Reparametrization.) Recall the change of variables formula for one variable: if $\phi\colon [a, b] \to \mathbb{R}$ is a continuously differentiable function and g is a continuous real-valued function defined on the image of ϕ, then $\int_{\phi(a)}^{\phi(b)} g = \int_a^b (g \circ \phi)\phi'$. Let $\psi\colon [a, b] \to \mathbb{R}$ be a continuously differentiable function with $\psi'(t) > 0$ for all t. Let $C\colon [\psi(a), \psi(b)] \to \mathbb{R}^n$ be a continuously differentiable curve, and let f be a real-valued function defined on the image of C. Let $D := C \circ \psi$, another curve in $\mathbb{R}^n$.
   (a) Prove $\int_C f = \int_D f$. In your solution, be explicit in your use of the change of variables formula by writing $g = blah$ and $\phi = blah$; also note where you use the chain rule.
   (b) Make up your own example illustrating $\int_C f = \int_D f$.
   (c) What happens if, instead, $\psi'(t) < 0$ for all t? (No proof necessary.) Give your own example.

3. Calculate the following line integrals, $\int_C f\, dC$.
   (a) $C(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$, $f(x, y) = x + y$.
   (b) $C(t) = (t, 2t, 3t, 4t)$, $t \in [0, 1]$, $f(x, y, z, w) = x + y + z + w$.
   (c) $C(t) = (2t, t^2, \ln t)$, $t \in [1, 2]$, $f(x, y, z) = 1$ (find the length of the curve).

4. Find the flow of the following vector fields along the given curves.
   (a) $C(t) = (t, t^3)$, $t \in [0, 1]$, $F(x, y) = (x, y)$.
   (b) $C(t) = (t, 0)$, $D(t) = (1, t)$, $t \in [0, 1]$, $F(x, y) = (x, y)$. Find the flow along the chain $E = C + D$.
   (c) Note that the solutions to the last two problems are the same. It turns out that there is a potential function for F. Find one and show that the integral in the last two problems is just giving the change in potential.
   (d) A particle is traveling along the circular helix $C(t) = (\cos t, \sin t, t)$ subject to the force $F(x, y, z) = (x, z, -yx)$. Find the total work done on the particle by the force for $0 \le t \le 3\pi$.

(e) $F(x, y) = e^x\vec{i} + e^y\vec{j}$ (guess what the notation means), and C is any parametrization of the portion of the ellipse $x^2 + 4y^2 = 4$ running clockwise from $(0, 1)$ to $(2, 0)$.

5. Find a potential function for $F(x, y) = (3x^2y + 1, x^3)$. If $C(t)$ is any curve connecting $(0, 0)$ to $(1, 1)$, what is the flow of F along C?

6. Does $F(x, y) = (-y, x)$ have a potential function? (Justify your answer, of course.)