HW 6, due Friday, March 29 *Math 212*

1. What is the volume of the cored apple:

$$V := \{(x, y, z) : r^2 \le x^2 + y^2 + z^2 \le R^2\}$$

   where $0 < r < R$?

2. Let $\Phi(u, v) = (u + v, u^2, 4v)$ and $\omega = z\,dx \wedge dy + y\,dx \wedge dz$. Compute the pullback $\Phi^*\omega$ and express it in standard form.

3. Let $\omega = f\,dx + g\,dy$ where $f(x, y) = \phi(x)$ and $g(x, y) = \psi(y)$, i.e., f depends only on x and g depends only on y. Let $\gamma\colon [a, b] \to \$R^2$ be any closed curve, i.e., $\gamma(a) = \gamma(b)$. Show that $\int_\gamma \omega = 0$.

4. (a) Let $\Phi : [0, 1]^2 \to \mathbb{R}^n$ be a 2-surface in $\mathbb{R}^n$. Show that $\partial^2\Phi = 0$ directly from the definition of the boundary map. (In fact, $\partial^2 = 0$ for k-surfaces in general, but you are just being asked to show this for 2-surfaces, to ease notation. A hint for the solution: $\partial^2\Phi$ will consist of sums of functions of the form $\Phi \circ \Delta^2_{i,\alpha} \circ \Delta^1_{1,\beta}$. The domain of all of these functions is $\mathbb{R}^0$, so to specify these functions, you need to tell me where the single point () comprising $\mathbb{R}^0$ is sent in each case.)

   (b) Let C be a k-chain in $\mathbb{R}^n$, and let $\omega \in \Omega^{k-2}\mathbb{R}^n$. Since direct calculation shows that $\partial^2 = 0$, we have that $\int_{\partial^2 C} \omega = 0$. Instead, without assuming $\partial^2 = 0$, prove $\int_{\partial^2 C} \omega = 0$ using Stokes' theorem.

5. Let

$$\begin{aligned} \Phi : [0, 1]^2 &\to \mathbb{R}^3 \\ (u, v) &\mapsto (u, v, u^2 + v^2), \end{aligned}$$

   and let $\omega = (x^2 - z)\,dx + (x + y)\,dy + (y + z)\,dz$.

   (a) Compute each $\Phi \circ \Delta^2_{i,\alpha}$.

   (b) Verify Stokes' theorem: $\int_{\partial\Phi} \omega = \int_\Phi d\omega$.

(Don't forget the next page.)

6. (a) Let

$$\begin{aligned} \Phi : [0,1] &\to \mathbb{R}^3 \\ t &\mapsto (t, t^2, t^3), \end{aligned}$$

and let $\omega = xy + z^2 \in \Omega^0\mathbb{R}^3$, a 0-form in $\mathbb{R}^3$. Verify Stokes' theorem.

(b) Let f be an $\mathbb{R}$-valued function of one variable. Let $a \leq b$ and define $\Phi(t) = a + (b-a)t$ for $0 \leq t \leq 1$ (so that Φ parametrizes the interval $[a, b]$). Let $\omega = f \in \Omega^0\mathbb{R}$. Verify Stokes' theorem by appealing to a result from one-variable calculus. (Make sure to name the relevant result.)