**Proposition.** Let $f, g\colon B \to \mathbb{R}$ be bounded functions on some subset $B \subseteq \mathbb{R}^n$. Then

$$\sup\{f(x)+g(x) : x \in B\} \leq \sup\{f(x) : x \in B\} + \sup\{g(x) : x \in B\},$$
$$\inf\{f(x)+g(x) : x \in B\} \geq \inf\{f(x) : x \in B\} + \inf\{g(x) : x \in B\}.$$

PROOF: We will only prove the result about sups, the result for infs being similar. First note that since $f$ and $g$ are bounded on $B$, then so is $f+g$. Therefore, all the sups in the statement exist. To save space, we will use the notation

$$\begin{aligned} M(f) &= \sup\{f(x) : x \in B\}, \\ M(g) &= \sup\{g(x) : x \in B\}, \\ M(f+g) &= \sup\{f(x)) + g(x) : x \in B\}. \end{aligned}$$

We are trying to show $M(f+g) \leq M(f) + M(g)$. Let $y \in B$. Then $f(y) \leq M(f)$ and $g(y) \leq M(g)$, since the sup of a set is an upper bound for the set. Adding these inequalities gives

$$f(y) + g(y) \leq M(f) + M(g),$$

and this inequality holds for all $y \in B$. Therefore, $M(f) + M(g)$ is an upper bound for $\{f(x) + g(x) : x \in B\}$. Since $M(f+g)$ is the *least* upper bound for that set, we have $M(f+g) \leq M(f) + M(g)$, as required. □

**Example.** Here is an example that shows that strict inequality in the above proposition may hold. Define $f, g\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \neq -1 \\ 1 & \text{if } x = -1, \end{cases} \qquad g(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1. \end{cases}$$

Then

$$\sup\{f(x)+g(x) : x \in \mathbb{R}\} = f(1) + g(1) = 0 + 1 = 1$$

while

$$\sup\{f(x) : x \in B\} + \sup\{g(x) : x \in B\} = 1 + 1 = 2.$$