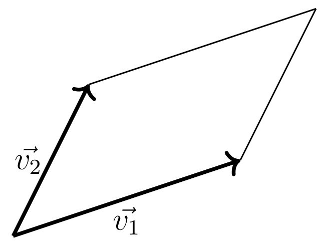
Geometric interpretation of the determinant

In this worksheet we will explore the geometric meaning of the determinant when working over $\mathbb{R}$.

Let $\vec{v_1} = (x_1, y_1)$ and $\vec{v_2} = (x_2, y_2)$ be linearly independent vectors in $\mathbb{R}^2$, and consider the parallelogram determined by them, as in the picture below.



Let $A(\vec{v_1}, \vec{v_2})$ denote the area of this parallelogram. Note that if $\vec{v_1}$ and $\vec{v_2}$ are linearly dependent, they determine what we might call a degenerate parallelogram whose area can be considered to be 0. Thus, we have defined a function

$$A\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}.$$

1. Let $k \in \mathbb{R}$. What is $A(k\vec{v_1}, \vec{v_2})$? (Be careful with the case $k < 0$.)

2. What is $A(\vec{v_1} + \vec{v_1'}, \vec{v_2})$? (You can do this by drawing the correct picture.)

3. What is $A(\vec{v}, \vec{v})$?

4. What is $A((1, 0), (0, 1))$?

5. You may have noted by now that the function A almost behaves like a determinant function (i.e., multilinear, alternating and normalized). What is the issue?

6. We can define the *signed area* of the parallelogram determined by $(\vec{v_1}, \vec{v_2})$ as follows. For this definition, the order of $\vec{v_1}$ and $\vec{v_2}$ matters. If $\vec{v_1}$ and $\vec{v_2}$ are linearly independent, they are both nonzero and non-parallel. Let θ be the angle from $\vec{v_1}$ to $\vec{v_2}$, measured counterclockwise (in radians, and between 0 and 2π). The signed area SA is defined as

$$SA(\vec{v_1}, \vec{v_2}) = \begin{cases} A(\vec{v_1}, \vec{v_2}) & \text{if } \theta < \pi \\ -A(\vec{v_1}, \vec{v_2}) & \text{if } \theta > \pi. \end{cases}$$

Prove that $SA(\vec{v_1}, \vec{v_2}) = \det(\vec{v_1}, \vec{v_2})$, where the latter means take the determinant of the 2×2 matrix with rows as given.

Hint: You can prove this by proving that SA is multilinear, alternating and normalized.

(Note that this formula also works if we take the matrix with *columns* given by $\vec{v_1}$ and $\vec{v_2}$.)

7. Consider the linear transformation $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ given by multiplication by the matrix $\begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}$.

   (a) What happens to the square with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$ under this transformation. How does the area change?

   (b) What happens to the square with vertices $(1, 1)$, $(2, 1)$, $(1, 2)$, $(2, 2)$?

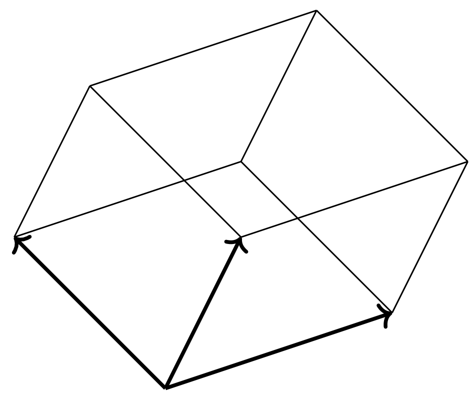
   (c) What happens to an arbitrary parallelogram on $\mathbb{R}^2$? In particular, how does the area change?

8. Let $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by multiplication by an arbitrary 2×2 matrix A (in our notation, $f = L_A$). Generalize the previous exercise to this case. Try to prove your statements.

Let $(\vec{v_1}, \ldots, \vec{v_n})$ be an n-tuple of vectors in $\mathbb{R}^n$. The *parallelepiped* formed by $(\vec{v_1}, \ldots, \vec{v_n})$ is the set

$$\{t_1\vec{v_1} + \ldots t_n\vec{v_n} \mid t_1, \ldots, t_n \in [0,1]\}.$$

When $n = 2$, this gives precisely the parallelogram we have been considering. In $\mathbb{R}^3$, we get a solid prism as long as the vectors are linearly independent.



We define the volume of a parallelepiped determined by $(\vec{v_1}, \ldots, \vec{v_n})$ as the absolute value of the determinant of the $n \times n$ matrix whose columns are given by those vectors.

9. Using properties of the determinant and your intuition about how a volume should behave, argue why this definition makes sense.

10. Consider the linear transformation $f\colon \mathbb{R}^n \to \mathbb{R}^n$ given by multiplication by the $n \times n$ matrix A. How does the volume of a box change under this transformation? First consider the case in which one of the vertices of the box is at the origin, and then generalize to an arbitrary box.