Math 201 lecture for Wednesday, Week 13

Cross product

Let $v_1, \dots, v_{n-1}$ be a set of $n-1$ vectors in $\mathbb{R}^n$. Define the function

$$\begin{aligned} \chi\colon \mathbb{R}^n &\to \mathbb{R} \\ x &\mapsto \det(x, v_1, \dots, v_{n-1}). \end{aligned}$$

where we think of the determinant as a function of the rows $x, v_1, \dots, v_{n-1}$ of a matrix, as usual. The $1 \times n$ matrix representing χ has the form $(a_1 \cdots a_n)$. We define the *cross product* to be the row vector

$$v_1 \times \cdots \times v_{n-1} := (a_1, \dots, a_n).$$

The mapping χ is just dot product with the cross product:

$$\chi(x) = (a_1 \cdots a_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (a_1, \dots, a_n) \cdot x = (v_1 \times \cdots \times v_{n-1}) \cdot x.$$

matrix multiplication   dot product

Theorem. (Properties of the cross product.)

(a) The cross product is a multilinear alternating function of $v_1, \dots, v_{n-1}$.

(b) Swapping v_i with v_j for $i \neq j$ changes the sign of the cross product.

(c) Adding a scalar multiple of v_i to v_j for some $i \neq j$ does not change the cross product.

(d) The cross product is orthogonal to the subspace spanned by $v_1, \dots, v_{n-1}$.

(e) The length of the cross product is the volume of the parallelepiped spanned by $v_1, \dots, v_{n-1}$.

(f) Given $w \in \mathbb{R}^n$, the volume of the parallelepiped spanned by w and $v_1, \dots, v_{n-1}$ is $|w \cdot (v_1 \times \cdots \times v_{n-1})|$.

(g) Let A be the $(n-1) \times n$ matrix with rows $v_1, \dots, v_{n-1}$, and let $A^{(j)}$ be the $(n-1) \times (n-1)$ matrix formed by removing the j-th column of A. Then

$$v_1 \times \cdots \times v_{n-1} = \left(\det(A^{(1)}), -\det(A^{(2)}), \det(A^{(3)}), \dots, (-1)^{n-1}\det(A^{(n)})\right).$$

Proof. Properties (a)–(c) follow immediately from the properties of $\det(x, v_1, \dots, v_{n-1})$. For property (d), note that

$$(v_1 \times \cdots \times v_{n-1}) \cdot v_i = \det(v_i, v_1, \dots, v_{n-1}) = 0$$

since v_i is a repeated row.

For property (e), let P be the parallelepiped spanned by $v_1, \dots, v_{n-1}$, and let Q be the parallelepiped spanned by $v_1 \times \cdots \times v_{n-1}$ and $v_1, \dots, v_{n-1}$. Since $v_1 \times \cdots \times v_{n-1}$ is perpendicular to P, the volume of Q is given by the volume of the base, P, times the height $\|v_1 \times \cdots \times v_{n-1}\|$:

$$\operatorname{vol}(Q) = \|v_1 \times \cdots \times v_{n-1}\| \operatorname{vol}(P). \tag{1}$$

The volume of Q is the absolute value of the determinant of its spanning vectors. Therefore,

$$\begin{aligned}\mathrm{vol}(Q) &= |\det(v_1 \times \cdots \times v_{n-1}, v_1, \ldots, v_{n-1})| \\ &= |\chi(v_1 \times \cdots \times v_{n-1}, v_1, \ldots, v_{n-1})| \\ &= (v_1 \times \cdots \times v_{n-1}) \cdot (v_1 \times \cdots \times v_{n-1}) \\ &= \|v_1 \times \cdots \times v_{n-1}\|^2.\end{aligned}$$

Combining this with equation (1) yields the result:

$$\|v_1 \times \cdots \times v_{n-1}\| = \mathrm{vol}(P).$$

For property (f), note that

$$|w \cdot (v_1 \times \cdots \times v_{n-1})| = |\det(w, v_1, \ldots, v_{n-1})|,$$

which gives the volume of the parallelepiped in question.

Property (g) follows by expanding the determinant defining χ along its first row:

$$\begin{aligned}\chi(x) &= \det(x, v_1, \ldots, v_{n-1}) \\ &= \det(A^{(1)}x_1 - \det(A^{(2)})x_2 + \cdots + (-1)^{n-1}\det(A^{(n)})x_n \\ &= (\det(A^{(1)}, -\det(A^{(2)}), \ldots, (-1)^{n-1}\det(A^{(n)})) \cdot (x_1, \ldots, x_n).\end{aligned}$$

□

The cross product in $\mathbb{R}^3$. The cross product is most well-known in the case $n = 3$. Here, we have vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The cross product is

$$x \times y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1) \in \mathbb{R}^3.$$

The usual mnemonic is

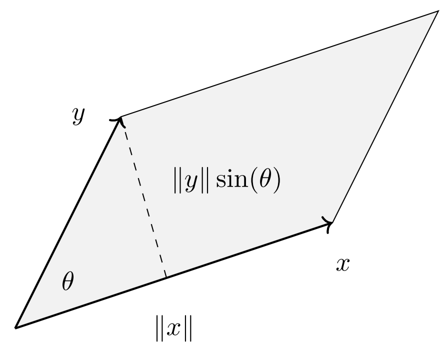
$$x \times y = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} = (x_2y_3 - x_3y_2)\mathbf{i} - (x_1y_3 - x_3y_1)\mathbf{j} + (x_1y_2 - x_2y_1)\mathbf{k},$$

where $\mathbf{i} = e_1 = (1, 0, 0)$, $\mathbf{j} = e_2 = (0, 1, 0)$, and $\mathbf{k} = e_3 = (0, 0, 1)$. We get exactly the formula given by part (g) of the Theorem. The above is only a mnemonic since we have not defined a determinant in the case where the entries are vectors of various dimensions.

The cross product here is perpendicular to the parallelogram spanned by x and y, and its length is

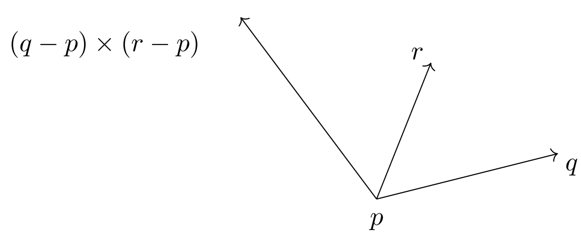
$$\|x \times y\| = \|x\|\|y\| \sin(\theta)$$

where θ is the angle between x and y. This last formula gives the area of the parallelogram spanned by x and y:



Example. Find an equation for the plane through the points $p = (1, 2, 3)$, $q = (1, 0, -2)$, and $r = (0, 7, 2)$.

SOLUTION: To find a vector perpendicular to the plane, we take the cross product of $q - p$ and $r - p$. Below is a picture that illustrates the geometry (with no attempt to get the actual coordinates correct!). The sides of the base parallelogram are spanned by the vectors $q - p$ and $r - p$.



Compute:

$$\begin{aligned}(q-p)\times(r-p) &= (0,-2,-5)\times(-1,5,-1)\\ &= \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & -5 \\ -1 & 5 & -1 \end{pmatrix}\\ &= 27\,\mathbf{i} + 5\,\mathbf{j} - 2\,\mathbf{k}\\ &= (27, 5, -2).\end{aligned}$$

To double-check, note that the cross product is perpendicular to $q - p$ and $r - p$:

$$(0,-2,-5)\cdot(27,5,-2) = 0 \quad \text{and} \quad (-1,5,-1)\cdot(27,5,-2) = 0.$$

The set of all points (x, y, z) perpendicular to the cross product is the plane defined by

$$(27, 5, -2)\cdot(x, y, z) = 0,$$

i.e., the plane with equation

$$27x + 5y - 2z = 0.$$

This plane passes through the origin, $(0,0,0)$. We want the translation of this plane that passes through p. (It will automatically then pass through q and r. So we could choose either q or r for this requirement, instead.) The equation of this translated plane will have the form

$$27x + 5y - 2z = c.$$

for some constant c. Plug in p (or q or r) to solve for c:

$$c = 27(1) + 5(2) - 2(3) = 31.$$

So the equation of the plane is

$$27x + 5y - 2z = 31.$$

(Check that the equation is satisfied by p, q, and r!)

Parametric equation of the plane. As we saw earlier in the semester, we can parametrize this plane by

$$\begin{aligned} f(s,t) &= p + s(q-p) + t(r-p) \\ &= (1,2,3) + s(0,-2,-5) + t(-1,5,-1) \\ &= (1-t, 2-2s+5t, 3-5s-t). \end{aligned}$$

Thus, we get the function:

$$\begin{aligned} f\colon \mathbb{R}^2 &\to \mathbb{R}^3 \\ (s,t) &\mapsto (1-t, 2-2s+5t, 3-5s-t). \end{aligned}$$

The image of f is the plane passing through p, q, and r. One may check that if we let

$$x = 1-t, \quad y = 2-2s+5t, \quad z = 3-5s-t,$$

then $27x + 5y - 2z = 31$, i.e., the point satisfies the equation for the plane.