Math 201 lecture for Monday, Week 11

Walks on graphs

We have devoted a lot of energy to the problem of diagonalizing a matrix. One major motivation for diagonalization is that it makes taking powers of a matrix easier. Explicitly, suppose that $A \in M_{n\times n}(F)$ is diagonalizable. So there exists a matrix P such that

$$P^{-1}AP = D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n).$$

It is easy to take powers of a diagonal matrix: $D^\ell = \operatorname{diag}(\lambda_1^\ell, \ldots, \lambda_n^\ell)$. Here is the important trick:

$$\begin{aligned} D^\ell &= (P^{-1}AP)^\ell \\ &= (P^{-1}AP)(P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP)(P^{-1}AP) \\ &= P^{-1}A(PP^{-1})A(PP^{-1})A(PP^{-1})\cdots(PP^{-1})AP \\ &= P^{-1}A^\ell P. \end{aligned}$$

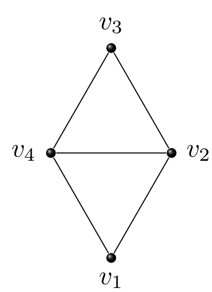
Therefore,

$$A^\ell = PD^\ell P^{-1} = P\operatorname{diag}(\lambda_1^\ell, \ldots, \lambda_n^\ell)P^{-1}.$$

In general, there will be many fewer arithmetic operations required on the right-hand side of this equation than on the left-hand side.

This lecture will consider one application of this idea.

Walks in graph. A graph (or network) consists of vertices connected by edges. Here is an example with 4 vertices connected by 5 edges:



The diamond graph.

A *walk of length* ℓ in a graph is a sequence of vertices $u_0u_1\ldots u_\ell$ where u_{i-1} is connected to u_i for $i = 1, \ldots, \ell$. So the length is the number of edges traversed. In our example, the following are walks from v_1 to v_4:

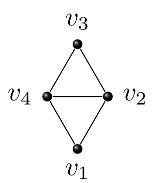
$$v_1v_4 \quad \text{and} \quad v_1v_2v_3v_4.$$

The first has length 1 and the second has length 3. We are interested in counting the number of closed walks between vertices.

Definition. Let G be a graph with vertices $v_1, \ldots, v_n$. The *adjacency matrix* of G is the $n \times n$ matrix $A = A(G)$ defined by

$$A_{ij} = \begin{cases} 1 & \text{if there is an edge connecting } v_i \text{ and } v_j \\ 0 & \text{otherwise.} \end{cases}$$

For example, the adjacency matrix of the diamond graph is



$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array}.$$

Theorem. Let A be the adjacency matrix for a graph G with vertices $v_1, \ldots, v_n$, and let $\ell \in \mathbb{Z} \geq 0$. Then then number of walks of length ℓ from v_i to v_j is $(A^\ell)_{ij}$.

Proof. Homework. □

Example. Consider the diamond graph and its adjacency matrix A, displayed above. Then

$$A^0 = I_4, \quad A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 2 & 5 & 2 & 5 \\ 5 & 4 & 5 & 5 \\ 2 & 5 & 2 & 5 \\ 5 & 5 & 5 & 4 \end{pmatrix}.$$

The highlighted entries in the matrix say there is 1 path of length 2 from v_2 to v_3 and there are 4 paths of length 3 from v_2 to itself. Can you find them? (The answer appears at the end of this lecture.)

So to count the number of walks, we need to compute powers of the adjacency matrix. Here is some good news:

Theorem. If A is an $n \times n$ symmetric matrix ($A = A^t$) over the real numbers, then it is diagonalizable over $\mathbb{R}$.

Proof. We may prove this later in the semester. (To look it up online, search for the "spectral theorem", which is usually stated for the more general class of Hermitian matrices. Over the real numbers, the Hermitian matrices are exactly the symmetric matrices.) □

This means that we can find a matrix P such that $P^{-1}AP = D$, where D is the diagonal matrix of the eigenvalues. Then $A^\ell = PD^\ell P^{-1}$. So we can find a nice closed form for the number of walks of length ℓ between any two vertices as a linear expression in the ℓ-th powers of the eigenvalues of A. If the eigenvalues are $\lambda_1, \ldots, \lambda_n$, the equation $A^\ell = PD^\ell P^{-1}$ immediately implies that for each pair of vertices v_i and v_j there exist real numbers $c_1, \ldots, c_n$, independent of ℓ, such that the number of closed walks of length ℓ from v_i to v_j is

$$c_1\lambda_1^\ell + \cdots + c_n\lambda_n^\ell.$$

The special case of closed walks is particularly nice.

Definition. A walk is *closed* if it begins and ends at the same vertex.

Definition. Let A be any $n \times n$ matrix. Then the *trace* of A is the sum of its diagonal entries:

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

Proposition. Let A be the adjacency matrix of a graph G. Then the number of closed walks in G of length ℓ is $\mathrm{tr}(A^\ell)$.

Proof. For each $i = 1, \ldots, n$, the number of closed walks from v_i to v_i is $(A^\ell)_{ii}$. Summing over i gives the total number of closed walks. □

Proposition. Let A be any $n \times n$ matrix. Then the trace of A is the sum of its eigenvalues, each counted according to its (algebraic) multiplicity.

Proof. Homework. □

Corollary. Let A be the adjacency matrix of a graph G with n vertices, and let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ be its list of (not necessarily distinct) eigenvalues. Then the number of closed walks in G of length ℓ is $\sum_{i=1}^{n} \lambda_i^\ell$.

Proof. The number of closed walks of length ℓ is $\mathrm{tr}(A^\ell)$, which is the sum of the eigenvalues of A^ℓ. By homework (an easy induction argument), if λ is an eigenvalue of A, then λ^ℓ is an eigenvalue of A^ℓ with unchanged eigenspace. It follows that the eigenvalues for A^ℓ are $\lambda_1^\ell, \ldots, \lambda_n^\ell$. □

Example. Let A be the adjacency matrix of the diamond graph G. The characteristic polynomial of A is

$$\det(A - xI_4) = x^4 - 5x^2 - 4x = x(x+1)(x^2 - x - 4).$$

Using the quadratic equation, we find the eigenvalues for A:

$$0, -1, \frac{1+\sqrt{17}}{2}, \frac{1-\sqrt{17}}{2}.$$

Therefore, the number of closed walks in G of length ℓ is

$$w(\ell) = (0)^\ell + (-1)^\ell + \left(\frac{1+\sqrt{17}}{2}\right)^\ell + \left(\frac{1-\sqrt{17}}{2}\right)^\ell,$$

where

$$(0)^\ell = \begin{cases} 1 & \text{if } \ell = 0 \\ 0 & \text{if } \ell > 0. \end{cases}$$

The following table gives the number of closed walks for $\ell = 0, 1, \ldots, 6$:

ℓ	0	1	2	3	4	5	6
$w(\ell)$	4	0	10	12	50	100	298

Exercise. The complete graph, K_n, has vertices $1, \ldots, n$ and an edge between every pair of vertices. How many closed walks are there in K_n of lenght ℓ?

Questions.

(a) How would you generalize today's ideas to the case of a directed graph (in which the edges have directions)?

(b) How would you generalize today's ideas to the case in which the edges have weights? (A special case would be to let the weight of edge (u, v) be the probability that the edge is traversed given that the starting point is u. Another possibility is to think of the weight as a cost for traveling across the edge.)

Answer to example on page 2: $v_2v_4v_3$ has length 2 and the following have length 3: $v_2v_3v_4v_2$, $v_2v_4v_3v_2$, $v_2v_1v_4v_2$, and $v_2v_4v_1v_2$.