Math 201 lecture for Friday, Week 11

Lengths, distances, components, angles

Definition. Let $(V, \langle\ ,\ \rangle)$ be an inner product space over $F = \mathbb{R}$ or $\mathbb{C}$. The *norm* or *length* of $x \in V$ is

$$\|x\| = \sqrt{\langle x, x\rangle} \in \mathbb{R}_{\geq 0}.$$

Two vectors $x, y \in V$ are *orthogonal* or *perpendicular* if $\langle x, y\rangle = 0$. A *unit vector* is a vector of norm 1: so $x \in V$ is a unit vector if $\|x\| = 1$, which is equivalent to $\langle x, x\rangle = 1$.

Examples of norms.

- $V = \mathbb{R}^n$, $\langle x, y\rangle = x \cdot y$, the usual dot product. Then for $x \in \mathbb{R}^n$,

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

- $V = \mathbb{C}^n$, $\langle x, y\rangle = x \cdot \overline{y}$, the usual dot product on $\mathbb{C}^n$. Then for $z \in \mathbb{C}^n$,

$$\begin{aligned}\|z\| &= \sqrt{z_1\overline{z_1} + \cdots + z_n\overline{z_n}}\\ &= \sqrt{|z_1|^2 + \cdots + |z_n|^2}.\end{aligned}$$

  If $z_j \in \mathbb{C}$ is written as $z_j = x_j + iy_j$ with $x_j, y_j \in \mathbb{R}$, then $|z_j|^2 = x_j^2 + y_j^2$. So then

$$\|z\| = \sqrt{x_1^2 + y_1^2 + \cdots + x_n^2 + y_n^2}.$$

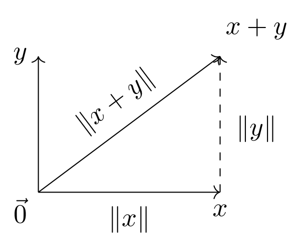
  Thus, if we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via the isomorphism

$$(x_1 + iy_1, \ldots, x_n + iy_n) \to (x_1, y_1, \ldots, x_n, y_n),$$

  then the isomorphism preserves norms.

Proposition. (Pythagorean theorem) Let $(V, \langle\ ,\ \rangle)$ be an inner product space over $F = \mathbb{R}$ or $\mathbb{C}$, and let $x, y \in V$ be perpendicular. Then

$$\|x\|^2 + \|y\|^2 = \|x + y\|^2.$$



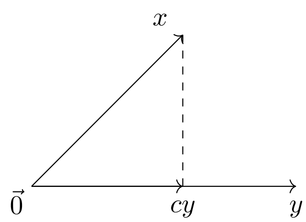
Proof. Since x and y are perpendicular, we have $\langle x, y\rangle = 0$. It follows that $\langle y, x\rangle = \overline{\langle x, y\rangle} = 0$, too. Therefore,

$$\|x + y\|^2 = \langle x + y, x + y\rangle$$

$$
\begin{aligned}
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\
&= \langle x, x\rangle + \langle y, y\rangle \\
&= \|x\|^2 + \|y\|^2.
\end{aligned}
$$

□

Suppose we are given two vectors x, y in an inner product space. A useful geometric operation is to break x into two parts, one of which lies along the vector y. Given any number c, the vector cy lies along y and we can evidently write x as the sum of two vectors: $x = (x - cy) + cy)$. In addition, though, we would like to require, by adjusting c, that the vector $x - cy$ is perpendicular to y. The picture in $\mathbb{R}^2$ would be:



We can calculate the required scalar c:

$$\langle x - cy, y\rangle = 0 \Longleftrightarrow \langle x, y\rangle - c\langle y, y\rangle = 0 \Longleftrightarrow c = \frac{\langle x, y\rangle}{\langle y, y\rangle} \Longleftrightarrow c = \frac{\langle x, y\rangle}{\|y\|^2},$$

which makes sense as long as $y \neq 0$.

Definition. Let $(V, \langle\ ,\ \rangle)$ be an inner product space over $F = \mathbb{R}$ or $\mathbb{C}$, and let $x, y \in V$ with $y \neq 0$. The *component* of x along y is the scalar

$$c = \frac{\langle x, y\rangle}{\langle y, y\rangle} = \frac{\langle x, y\rangle}{\|y\|^2}.$$

The *orthogonal projection* of x to y is the vector cy.

Example. Let $x \in V = \mathbb{R}^n$ or $\mathbb{C}^n$, and let e_j be the j-th standard basis vector. Then

$$\frac{\langle x, e_j\rangle}{\langle e_j, e_j\rangle} = \frac{x_j}{1} = x_j.$$

Thus, x_j is the component of x along e_j, and $x_j e_j$ is the projection of x to e_j.

Example. Let $x = (3, 2)$ and $y = (5, 0) = 5e_1$ in $\mathbb{R}^2$ with the usual inner product. Then the component of x along y is

$$\frac{\langle x, y\rangle}{\langle y, y\rangle} = \frac{(3, 2) \cdot (5, 0)}{(5, 0), (5, 0)} = \frac{15}{25} = \frac{3}{5}.$$

So the projection of x to y is

$$cy = \frac{3}{5}(5, 0) = (3, 0),$$

as expected.

Proposition. Let $(V, \langle\ ,\ \rangle)$ be an inner product space over $F = \mathbb{R}$ or $\mathbb{C}$. Let $x, y \in V$ and $c \in F$. Then

(a) $\|cx\| = |c|\|x\|$.

(b) $\|x\| = 0$ if and only if $x = 0$.

(c) Cauchy-Schwarz inequality: $|\langle x, y\rangle| \leq \|x\|\|y\|$.

(d) Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$.

Proof. Parts (a) and (b) are left as exercises. Part (c) is tricky. If $y = 0$, we're done. So assume $y \neq 0$, and let $c = \langle x, y\rangle / \langle y, y\rangle$ be the component of x along y. By construction, $x - cy$ is perpendicular to y and hence to cy. Therefore, by the Pythagorean theorem,

$$\|x - cy\|^2 + \|cy\|^2 = \|(x - cy) + cy\|^2 = \|x\|^2.$$

Since $\|x - cy\|^2 \geq 0$, if we drop that term in the above equation, we get

$$\|cy\|^2 \leq \|x\|^2.$$

Take square roots to get

$$\|x\| \geq \|cy\| = |c|\|y\| = \left|\frac{\langle x, y\rangle}{\|y\|^2}\right| \|y\| = \frac{|\langle x, y\rangle|}{\|y\|}.$$

Multiply through by $\|y\|$ to get Cauchy-Schwarz.

For the proof of the triangle inequality, we will need two basis results concerning complex numbers. Let $z = a + bi$ be any complex number. Then we have (i) $z + \overline{z} = (a + bi) + (a - bi) = 2a$. So

$$z + \overline{z} = 2\,\mathrm{Re}(z),$$

and (ii) $|z| = \sqrt{a^2 + b^2} \geq |a|$. So

$$\mathrm{Re}(z) \leq |z|.$$

The triangle inequality is then an easy consequence of Cauchy-Schwarz:

$$\begin{aligned}
\|x + y\|^2 &= \langle x + y, x + y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\
&= \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2 \\
&= \|x\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} + \|y\|^2 \\
&= \|x\|^2 + 2\,\mathrm{Re}(\langle x, y\rangle) + \|y\|^2 \\
&\leq \|x\|^2 + 2\,|\langle x, y\rangle| + \|y\|^2 \\
&\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\
&\leq (\|x\| + \|y\|)^2.
\end{aligned}$$

Take square roots to get the triangle inequality. □

Distance. Let $(V, \langle\ ,\ \rangle)$ be an inner product space over $\mathbb{R}$ or $\mathbb{C}$. The *distance* between $x, y \in V$ is defined to be

$$d(x, y) := \|x - y\|.$$

The following properties then easily follow from what we have already done:

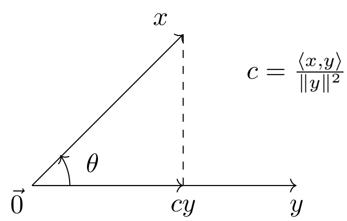
Proposition. For all $x, y, z \in V$,

(a) Symmetry: $d(x, y) = d(y, x)$.

(b) Positive-definiteness: $d(x, y) \geq 0$, and $d(x, y) = 0$ iff $x = y$.

(c) Triangle inequality: $d(x, y) \leq d(x, z) + d(z, y)$.

Angles. Now let $(V, \langle\ ,\ \rangle)$ be an inner product space over $F = \mathbb{R}$. (So we will not consider the case $F = \mathbb{C}$ in our discussion of angles.) We would like to define the notion of an *angle* between $x, y \in V$. Our picture for the component provides motivation:



The dashed vertical line and the vector y are perpendicular (by definition of c). The cosine of the displayed angle should be the length of the base, cy, divided by the length of the hypotenuse, x. That quotient is

$$\frac{\|cy\|}{\|x\|} = |c|\frac{\|y\|}{\|x\|} = \frac{|\langle x, y\rangle|}{\|y\|^2}\frac{\|y\|}{\|x\|} = \frac{|\langle x, y\rangle|}{\|x\|\|y\|}.$$

Omitting the absolute value on the real number $\langle x, y\rangle$ in the numerator provides the correct signs for the different quadrants (when θ is not between 0 and 90 degrees).

Definition. Let $(V, \langle\ ,\ \rangle)$ be an inner product space over $F = \mathbb{R}$, and let x, y be nonzero elements of V. The *angle* θ between x and y is

$$\theta = \cos^{-1}\left(\frac{\langle x, y\rangle}{\|x\|\|y\|}\right),$$

and thus,

$$\langle x, y\rangle = \|x\|\|y\|\cos(\theta).$$

Remarks.

- Cauchy-Schwarz says $|\langle x, y\rangle \leq \|x\|\|y\|$. Therefore,

  $$-1 \leq \frac{\langle x, y\rangle}{\|x\|\|y\|} \leq 1.$$

  So the inverse cosine in the definition of the angle always makes sense.

- In the definition of the angle, it might make more sense conceptually to write

  $$\cos(\theta) = \left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|} \right\rangle.$$

  In other words, the cosine of the angle between x and y is the inner product of their directions where the *direction* of a vector w is taken to be the scalar multiple of w with unit length, $w/\|w\|$.