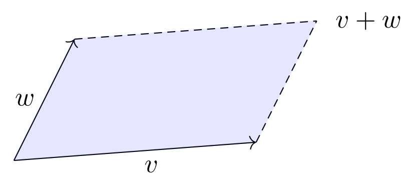
Math 201 lecture for Wednesday, Week 9

Determinants and volume

The *parallelogram spanned by* $v, w \in \mathbb{R}^2$ is

$$P = \{\lambda v + \mu w \colon \lambda, \mu \in [0,1]\}$$

where $[0,1] = \{x \in \mathbb{R} \colon 0 \leq x \leq 1\}$:



Theorem. Let $A(v,w)$ be the area of the parallelogram spanned by $v, w \in \mathbb{R}^2$. Then

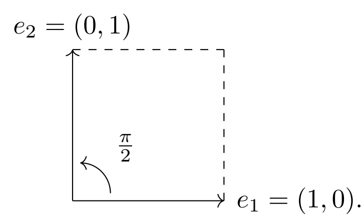
$$A(v,w) = |\det(v,w)|,$$

where $\det(v,w)$ is the determinant of the matrix with rows v and w.

Note: Since the determinant of a square matrix and its transpose are the same, $A(v,w)$ is also the absolute value of the matrix whose *columns* are v and w.

Proof of theorem. Define $SA(v,w)$ to be the signed area defined in the worksheet. We show that SA satisfies the properties required of a determinant function. Then, since the determinant is unique, it follows that $SA(v,w) = \det(v,w)$ and the result follows since $A(v,w) = |SA(v,w)|$.

- **Normalized.** We have $SA(e_1, e_2) = 1$:

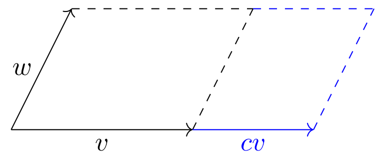


  The sign is positive since the angle from e_1 to e_2 is less than π.

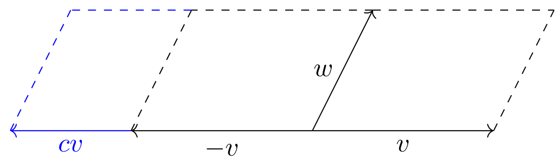
- **Alternating.** We have $SA(v,v) = 0$ since in this case, the corresponding parallelogram is degenerate.

- **Multilinear.**
  - $SA(cv, w) = cSA(v, w)$ and $SA(v, cw) = cSA(v, w)$:

    $c > 0$

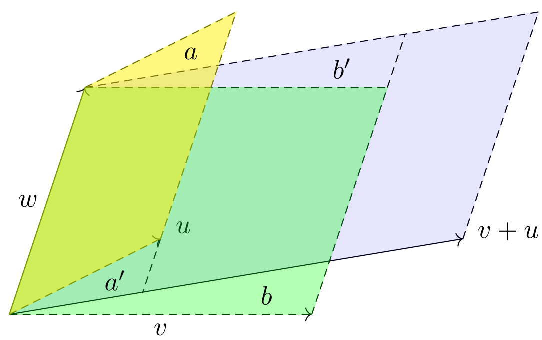
    

    $c < 0$

    The areas are scaled by $|c|$ in either case since the base is scaled by $|c|$ and the height does not change. The drawing assumes that the angle from v to w is less than π. There is a similar drawing for the case where the angle is greater than π. Either way, in the case where $c < 0$ note that although $SA(cv, w)$ and $SA(v, w)$ have opposite signs, $SA(cv, w)$ and $cSA(v, w)$ have the same sign.

    Similar drawings show that $SA(v, cw) = cSA(v, w)$.
  - $SA(v + u, w) = SA(v, w) + SA(u, w)$:

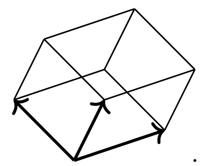
    Note how to dissect the u-w and v-w parallelograms to get the $(v + u)$-w parallelogram: Cut section a in the u-w parallelogram and place it at section a', then cut section b in the v-w parallelogram and place it at section b'. The result is two parallelograms that can exactly cover the $(v + u)$-w parallelogram.

    Of course, our drawing is just one case among the many possible angles between pairs of v, u, and w.

□

Definition. The *parallelepided spanned by* $v_1, \dots, v_n \in \mathbb{R}^n$ is

$$P = \{\lambda_1 v_1 + \cdots + \lambda_n v_n : \lambda_i \in [0,1] \text{ for } i = 1, \dots, n\},$$



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It turns out that the volume of P is given by the determinant of the matrix whose row (or columns) are $v_1, \dots, v_n$:

$$\text{vol}(P) = |\det(v_1, \dots, v_n)|.$$

Note that one of the vertices of P is the origin (set $\lambda_1 = \cdots = \lambda_n = 0$). To get an arbitrary parallelepiped in $\mathbb{R}^n$ we can just translate by any vector $u \in \mathbb{R}^n$:

$$P + u := \{p + u : p \in P\} = \{\lambda_1 v_1 + \cdots + \lambda_n v_n + u : 0 \leq \lambda_i \leq 1 \text{ for } i = 1, \dots, n\}.$$

The volume does not change:

$$\text{vol}(P + u) = |\det(v_1, \dots, v_n)|.$$

Theorem. Let P be the parallelepided spanned by $v_1, \dots, v_n \in \mathbb{R}^n$. Let $A \in M_{n\times n}(\mathbb{R})$, and let $L_A \colon \mathbb{R}^n \to \mathbb{R}^n$ be the corresponding linear function, $L_A(x) = Ax$. Then $L_A(P)$ is the parallelepiped spanned by the vectors $Av_1, \dots, Av_n$, and

$$\text{vol}(L_A(P)) = |\det(A)|\text{vol}(P).$$

Moreover, $L_A(P+u) = L_A(P) + L_A(u)$. Thus, *application of L_A scales the volumes of parallelepipeds in $\mathbb{R}^n$ be a factor of $|\det(A)|$.*

Proof. We have $x \in L_A(P)$ if and only if there exist $\lambda_1, \dots, \lambda_n \in [0,1]$ such that

$$x = A(\lambda_1 v_1 + \cdots + \lambda_n v_n) = \lambda_1 A v_1 + \cdots + \lambda_n A v_n,$$

i.e., if and only if x is in the parallelepiped determined by $Av_1, \dots, Av_n$.

Let B be the matrix with columns $v_1, \dots, v_n$. Then $\text{vol}(P) = |\det(B)|$, Note that AB is the matrix whose columns are $Av_1, \dots, Av_n$. It follows that

$$\text{vol}(L_A(P)) = |\det(AB)| = |\det(A)||\det(B)|.$$

We have $L_A(P+u) = L_A(P) + L_A(u)$ since L_A is linear. □

Remark. To approximate the volume of an arbitrary shape in $\mathbb{R}^n$, one can try to dissect the shape into a union of parallelepipeds. One definition of the volume of an arbitrary shape is derived by taking limits of such approximations. One can then ask how the volume of a shape S changes under the application of a function $f \colon \mathbb{R}^n \to \mathbb{R}^n$. If the function is "nice" (differentiable), then at each point p in the shape, one creates a linear approximation $Df(p)$ of the function f (called the *derivative of* f *at* p), akin to L_A, above. Further assuming that f is injective, the volume of the image of the shape is then given by the *change of variables* formula in multivariable calculus:

$$\mathrm{vol}(f(S)) = \int_{p \in S} |\det(Df(p))|.$$

In light of the theorem we just proved, the determinant $|Df(p)|$ should be thought of as a scaling factor. It tells us how much f scales volumes (infinitesimally) at p.