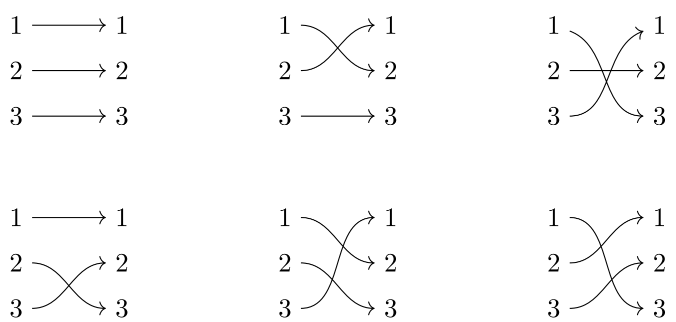
Math 201 lecture for Wednesday, Week 8

Permutation expansion of the determinant

Definition. A *permutation* of a set X is a bijective mapping of X to itself. If σ and τ are permutations of X, then so is their composition $\sigma \circ \tau$. The collection of all permutations of X along with the binary operation $\circ$ given by composition of functions is called the *symmetric group on* X. For each nonnegative integer n, let $[n] := \{1, \dots, n\}$. The symmetric group on $[n]$ is called the *symmetric group of degree* n and denoted by $\mathfrak{S}_n$.

Example. Here are six elements of $\mathfrak{S}_3$:



Note. Define the *factorial* of a natural number as follows: $0! = 1$, and for each integer $n > 0$, recursively define $n! = n(n-1)!$. Thus, $1! = 1$, $2! = 2 \cdot 1$, $3! = 3 \cdot 2 \cdot 1 = 6$, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$, etc. Then the number of elements of $\mathfrak{S}_n = n!$ since we can uniquely determine every permutation σ by first choosing one of n values for $\sigma(1)$, then any of the remaining values $n-1$ for $\sigma(2)$, then one of the remaining $n-2$ values of $\sigma(3)$, etc.

Definition. Let $\sigma \in \mathfrak{S}_n$. The *permutation matrix corresponding to* σ is the $n \times n$ matrix P_σ whose i-th row is $e_{\sigma(i)}$. Another way of saying this is that, P_σ is obtained by permuting the columns of the identity matrix, I_n, according to σ: put e_j in column $\sigma(j)$.

Example. Let $\sigma \in \mathfrak{S}_3$ be defined by $\sigma(1) = 2$, $\sigma(2) = 3$, and $\sigma(3) = 1$. Then

$$P_\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Exercise. Let $\sigma, \tau \in \mathfrak{S}_n$ and let A be an $n \times n$ matrix.

(a) If the rows of A are $(r_1, \dots, r_n)$, then the i-th row of $P_\sigma A$ is $r_{\sigma(i)}$. In other words, the multiplying on the left by P_σ permutes the rows of A in the same way that the rows of I_n are permuted to form P_σ. We leave it as an exercise to the reader to investigate the effect of multiplying A on the right by P.

(b) $P_\sigma e_{\sigma(i)} = e_i$, and $P_\sigma P_\tau = P_{\tau \circ \sigma}$. (Note that the order of σ and τ have switched.)

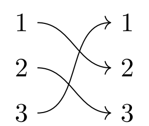
Rook placements. Permutation matrices are exactly those that have a single 1 in each row and in each column. Thus, if the 1s in a permutation matrix were replaced by rooks in the game of chess, then no rook would be attacking another. We sometimes call a permutation matrix a *rook placement*.

Definition. The *sign* of $\sigma \in \mathfrak{S}_n$ is

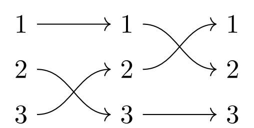
$$\operatorname{sign}(\sigma) = \det(P_\sigma) = \pm 1.$$

A permutation is *even* if its sign is 1 and *odd* if its sign is -1.

Every permutation matrix P_σ may be obtained from I_n through a sequence of transpositions of columns, i.e., a sequence in which each step consists of swapping two columns. In the example above, P_σ is formed by permuting columns 1, 2, and 3 of I_3 as follows:



This permutation could have been obtained from two transpositions:



Thus, every permutation matrix can be obtained as the product of permutation matrices corresponding to transpositions. Swapping two columns in a matrix changes the sign of the determinant. Therefore, even though a permutation σ may be realized in different ways as sequences of transpositions, the parity (evenness or oddness) of the number of transpositions required is well-defined: the number is even if $\det(P_\sigma) = 1$ and odd if $\det(P_\sigma) = -1$.

Theorem. Let A be an $n \times n$ matrix. Then

$$\det(A) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sign}(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{n\sigma(n)}.$$

Example. Consider the case $n = 3$. Then

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Each term $A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{n\sigma(n)}$ in the formula in the theorem should be thought of as the product of the entries corresponding to a rook placement. The permutations, rook placements, and corresponding summands appear in Figure 1.[1]

[1]Note that the sign of the permutation is also equal to $(-1)^c$ where c is the number of times two arrows cross in the diagram for the permutation in the left-most column.

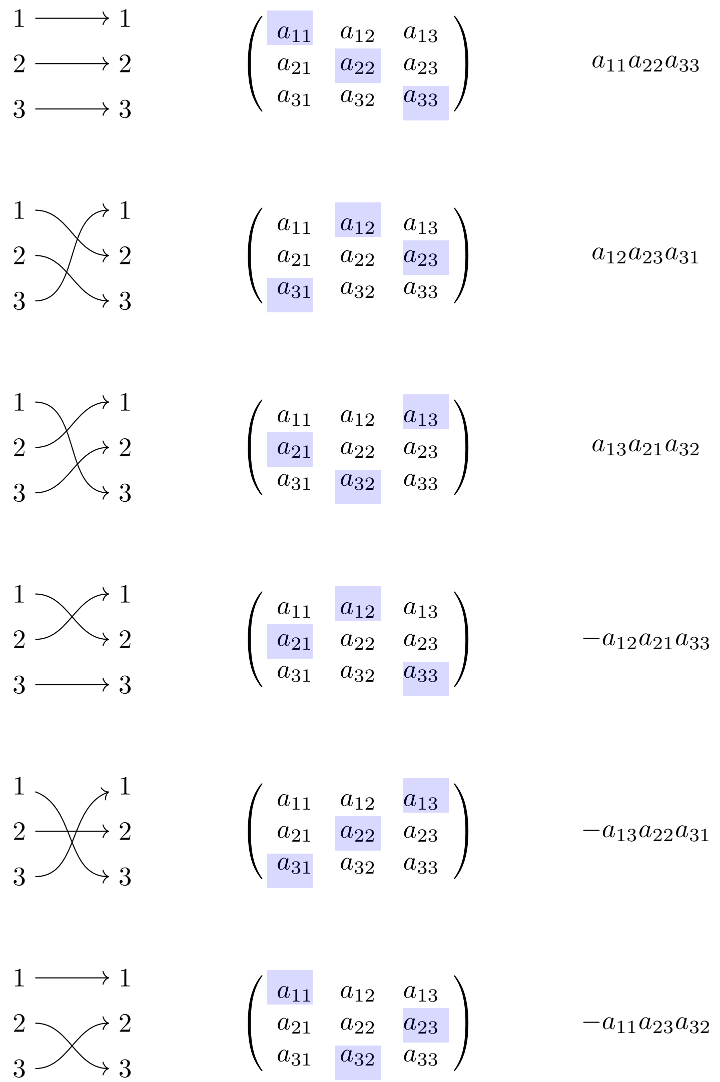


Figure 1: Computing the determinant of the 3×3 matrix $A = (a_{ij})$ via rook placements. The determinant is the sum of the terms in the right-most column.

Proof of permutation formula for the determinant. We want to compute

$$\det(A_{11}e_1 + A_{12}e_2 + \cdots + A_{1n}e_n, \quad \ldots \quad, A_{n1}e_1 + A_{n2}e_2 + \cdots + A_{nn}e_n).$$

Each of the n components in the above expression consists of n summands where each of the summands has the form $a_{ij}e_j$. Using the multilinear properties of the determinant, when we expand the above express, we get $n!$ terms, each of the form

$$A_{1j_1}A_{2j_2}\cdots A_{nj_n}\det(e_{1j_1}, e_{2j_2}, \ldots, e_{nj_n}).$$

If any pair of these e_{kj_k} is the same, this term will evaluate to 0. Thus, for the nonzero terms, $e_{1j_1}, e_{2j_2}, \ldots, e_{nj_n}$ must be some permutation of $e_1, \ldots, e_n$. We then have

$$\det(e_{1j_1}, e_{2j_2}, \ldots, e_{nj_n}) = \pm 1,$$

depending on the sign of the permutation σ defined by

$$\sigma(1) = j_1,\ \sigma(2) = j_2,\ \ldots\ ,\ \sigma(n) = j_n,$$

and we can write

$$A_{1j_1}A_{2j_2}\cdots A_{n_{j_n}}\det(e_{1j_1}, e_{2j_2}, \ldots, e_{nj_n}) = \operatorname{sign}(\sigma)A_{1\sigma(1)}A_{2\sigma(2)}\cdots A_{n\sigma(n)}.$$

□

See the next pages for all of the details of the above proof in the case $n = 3$.

Let's look at the proof again in the case $n = 3$. The i-th row vector of A is

$$r_i = a_{i1}e_1 + a_{i2}e_2 + a_{i3}e_3.$$

To compute the determinant of A we start by expanding using multilinearity:

$$\begin{aligned}
\det(A) &= \det(r_1, r_2, r_3) \\
&= \det(a_{11}e_1 + a_{12}e_2 + a_{13}e_3, a_{21}e_1 + a_{22}e_2 + a_{23}e_3, a_{31}e_1 + a_{32}e_2 + a_{33}e_3) \\
&= \det(a_{11}e_1, a_{21}e_1 + a_{22}e_2 + a_{23}e_3, a_{31}e_1 + a_{32}e_2 + a_{33}e_3) \\
&\quad + \det(a_{12}e_2, a_{21}e_1 + a_{22}e_2 + a_{23}e_3, a_{31}e_1 + a_{32}e_2 + a_{33}e_3) \\
&\qquad + \det(a_{13}e_3, a_{21}e_1 + a_{22}e_2 + a_{23}e_3, a_{31}e_1 + a_{32}e_2 + a_{33}e_3) \\
&= \det(a_{11}e_1, a_{21}e_1, a_{31}e_1 + a_{32}e_2 + a_{33}e_3) \\
&\quad + \det(a_{11}e_1, a_{22}e_2, a_{31}e_1 + a_{32}e_2 + a_{33}e_3) \\
&\qquad + \det(a_{11}e_1, a_{23}e_3, a_{31}e_1 + a_{32}e_2 + a_{33}e_3) \\
&\quad + \det(a_{12}e_2, a_{21}e_1, a_{31}e_1 + a_{32}e_2 + a_{33}e_3) \\
&\qquad + \det(a_{12}e_2, a_{22}e_2, a_{31}e_1 + a_{32}e_2 + a_{33}e_3) \\
&\qquad\quad + \det(a_{12}e_2, a_{23}e_3, a_{31}e_1 + a_{32}e_2 + a_{33}e_3) \\
&\qquad + \det(a_{13}e_3, a_{21}e_1, a_{31}e_1 + a_{32}e_2 + a_{33}e_3) \\
&\qquad\quad + \det(a_{13}e_3, a_{22}e_2, a_{31}e_1 + a_{32}e_2 + a_{33}e_3) \\
&\qquad\qquad + \det(a_{13}e_3, a_{23}e_3, a_{31}e_1 + a_{32}e_2 + a_{33}e_3)
\end{aligned}$$

There is one more step to go in the complete expansion, at which point, we'll have 27 terms. For completeness, I'll list these all on the next page.

$$
\begin{aligned}
&= \det(a_{11}e_1, a_{21}e_1, a_{31}e_1) \\
&\qquad + \det(a_{11}e_1, a_{21}e_1, a_{32}e_2) \\
&\qquad\qquad + \det(a_{11}e_1, a_{21}e_1, a_{33}e_3) \\
&\qquad + \det(a_{11}e_1, a_{22}e_2, a_{31}e_1) \\
&\qquad\qquad + \det(a_{11}e_1, a_{22}e_2, a_{32}e_2) \\
&\qquad\qquad\qquad + \det(a_{11}e_1, a_{22}e_2, a_{33}e_3) \\
&\qquad\qquad + \det(a_{11}e_1, a_{23}e_3, a_{31}e_1) \\
&\qquad\qquad\qquad + \det(a_{11}e_1, a_{23}e_3, a_{32}e_2) \\
&\qquad\qquad\qquad\qquad + \det(a_{11}e_1, a_{23}e_3, a_{33}e_3) \\
&+ \det(a_{12}e_2, a_{21}e_1, a_{31}e_1) \\
&\qquad + \det(a_{12}e_2, a_{21}e_1, a_{32}e_2) \\
&\qquad\qquad + \det(a_{12}e_2, a_{21}e_1, a_{33}e_3) \\
&\qquad + \det(a_{12}e_2, a_{22}e_2, a_{31}e_1) \\
&\qquad\qquad + \det(a_{12}e_2, a_{22}e_2, a_{32}e_2) \\
&\qquad\qquad\qquad + \det(a_{12}e_2, a_{22}e_2, a_{33}e_3) \\
&\qquad\qquad + \det(a_{12}e_2, a_{23}e_3, a_{31}e_1) \\
&\qquad\qquad\qquad + \det(a_{12}e_2, a_{23}e_3, a_{32}e_2) \\
&\qquad\qquad\qquad\qquad + \det(a_{12}e_2, a_{23}e_3, a_{33}e_3) \\
&+ \det(a_{13}e_3, a_{21}e_1, a_{31}e_1) \\
&\qquad + \det(a_{13}e_3, a_{21}e_1, a_{32}e_2) \\
&\qquad\qquad + \det(a_{13}e_3, a_{21}e_1, a_{33}e_3) \\
&\qquad + \det(a_{13}e_3, a_{22}e_2, a_{31}e_1) \\
&\qquad\qquad + \det(a_{13}e_3, a_{22}e_2, a_{32}e_2) \\
&\qquad\qquad\qquad + \det(a_{13}e_3, a_{22}e_2, a_{33}e_3) \\
&\qquad\qquad + \det(a_{13}e_3, a_{23}e_3, a_{31}e_1) \\
&\qquad\qquad\qquad + \det(a_{13}e_3, a_{23}e_3, a_{32}e_2) \\
&\qquad\qquad\qquad\qquad + \det(a_{13}e_3, a_{23}e_3, a_{33}e_3)
\end{aligned}
$$

Use linearity to pull out the constants:

$$
\begin{aligned}
= a_{11}a_{21}a_{31}\ \det(e_1, e_1, e_1) \\
+ a_{11}a_{21}a_{32}\ \det(e_1, e_1, e_2) \\
+ a_{11}a_{21}a_{33}\ \det(e_1, e_1, e_3) \\
\\
+ a_{11}a_{22}a_{31}\ \det(e_1, e_2, e_1) \\
+ a_{11}a_{22}a_{32}\ \det(e_1, e_2, e_2) \\
+ a_{11}a_{22}a_{33}\ \det(e_1, e_2, e_3) \\
\\
+ a_{11}a_{23}a_{31}\ \det(e_1, e_3, e_1) \\
+ a_{11}a_{23}a_{32}\ \det(e_1, e_3, e_2) \\
+ a_{11}a_{23}a_{33}\ \det(e_1, e_3, e_3) \\
\\
+ a_{12}a_{21}a_{31}\ \det(e_2, e_1, e_1) \\
+ a_{12}a_{21}a_{32}\ \det(e_2, e_1, e_2) \\
+ a_{12}a_{21}a_{33}\ \det(e_2, e_1, e_3) \\
\\
+ a_{12}a_{22}a_{31}\ \det(e_2, e_2, e_1) \\
+ a_{12}a_{22}a_{32}\ \det(e_2, e_2, e_2) \\
+ a_{12}a_{22}a_{33}\ \det(e_2, e_2, e_3) \\
\\
+ a_{12}a_{23}a_{31}\ \det(e_2, e_3, e_1) \\
+ a_{12}a_{23}a_{32}\ \det(e_2, e_3, e_2) \\
+ a_{12}a_{23}a_{33}\ \det(e_2, e_3, e_3) \\
\\
+ a_{13}a_{21}a_{31}\ \det(e_3, e_1, e_1) \\
+ a_{13}a_{21}a_{32}\ \det(e_3, e_1, e_2) \\
+ a_{13}a_{33}a_{21} \det(e_3, e_1, e_3) \\
\\
+ a_{13}a_{22}a_{31}\ \det(e_3, e_2, e_1) \\
+ a_{13}a_{22}a_{32}\ \det(e_3, e_2, e_2) \\
+ a_{13}a_{22}a_{33}\ \det(e_3, e_2, e_3) \\
\\
+ a_{13}a_{23}a_{31}\ \det(e_3, e_3, e_1) \\
+ a_{13}a_{23}a_{32}\ \det(e_3, e_3, e_2) \\
+ a_{13}a_{23}a_{33}\ \det(e_3, e_3, e_3)
\end{aligned}
$$

Now we use the alternating property of the determinant. If any row is repeated, the determinant is 0. Getting rid of those terms leaves:

$$\begin{aligned}\det(A) = a_{11}a_{22}a_{33}\ \det(e_1, e_2, e_3) \\ + a_{11}a_{23}a_{32}\ \det(e_1, e_3, e_2) \\ + a_{12}a_{21}a_{33}\ \det(e_2, e_1, e_3) \\ + a_{12}a_{23}a_{31}\ \det(e_2, e_3, e_1) \\ + a_{13}a_{21}a_{32}\ \det(e_3, e_1, e_2) \\ + a_{13}a_{22}a_{31}\ \det(e_3, e_2, e_1).\end{aligned}$$

Next notice that each determinant appearing above is the determinant of a permutation matrix. For instance, the term

$$a_{12}a_{23}a_{31}\ \det(e_2, e_3, e_1)$$

contains $\det(e_2, e_3, e_1)$, which is the determinant of the permutation matrix for the permutation $\sigma(1) = 2$, $\sigma(2) = 3$, and $\sigma(3) = 1$. We have

$$\begin{aligned}a_{12}a_{23}a_{31}\ \det(e_2, e_3, e_1) &= a_{1\sigma(1)}a_{2\sigma(2)}a_{3\sigma(3)}\det(P_\sigma) \\ &= a_{1\sigma(1)}a_{2\sigma(2)}a_{3\sigma(3)}\text{sign}(\sigma).\end{aligned}$$

In this way, the six terms in the sum can be expressed as follows:

$$\det(A) = \sum_{\sigma \in \mathfrak{S}_3} \det(P_\sigma)a_{1\sigma(1)}a_{2\sigma(2)}a_{3\sigma(3)}$$

$$= \sum_{\sigma \in \mathfrak{S}_3} \text{sign}(P_\sigma)a_{1\sigma(1)}a_{2\sigma(2)}a_{3\sigma(3)}.$$