Math 201 lecture for Monday, Week 8

Determinant of the transpose

Goal: Our goal for the day is to show that the determinant of a matrix and the determinant of the transpose of that matrix are equal. It then immediately follows that the determinant is not only a multilinear, alternating, normalized function of the *rows* of a matrix (by definition), but it is also a multilinear, alternating, normalized function of its *columns*. So one may use both row and column operations to compute the determinant.

Elementary matrices. An $n \times n$ matrix is called an *elementary matrix* if it is obtained from the identity matrix, I_n, through a single elementary row operation (scaling a row by a nonzero scalar, swapping rows, or adding one row to another).

Here is why elementary matrices are interesting: Let E be an $n \times n$ elementary matrix corresponding to some row operation and let A be any $n \times k$ matrix. Then EA is the matrix obtained from A by performing that row operation. Thus, you can perform row operations through multiplication by elementary matrices.

Example. Let

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 0 & -1 & 2 \\ 1 & 5 & 6 & 7 \end{pmatrix}.$$

To find the elementary matrix that will subtract 3 times the first row of A from the second row, we do that same operation to the identity matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{r_2 \to r_2 - 3r_1} \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =: E_1.$$

Multiplying by E_1 on the left performs the same elementary row operation on A:

$$E_1 A = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 0 & -1 & 2 \\ 1 & 5 & 6 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -6 & -10 & -10 \\ 1 & 5 & 6 & 7 \end{pmatrix}.$$

Next, let E_2 be the elementary matrix corresponding to subtracting the first row from the third:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{r_3 \to r_3 - r_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} =: E_2.$$

Multiplying $E_1 A$ on the left by E_2 performs the corresponding row operation:

$$E_2 E_1 A = E_2(E_1 A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -6 & -10 & -10 \\ 1 & 5 & 6 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -6 & -10 & -10 \\ 0 & 3 & 3 & 3 \end{pmatrix}.$$

Next, swap the second and third rows:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} =: E_3,$$

and, thus,

$$E_3E_2E_1A = E_3(E_2E_1A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -6 & -10 & -10 \\ 0 & 3 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 3 & 3 \\ 0 & -6 & -10 & -10 \end{pmatrix}.$$

To continue row reduction, we would now scale the second row by $\frac{1}{3}$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{r_2 \leftrightarrow r_2/3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} =: E_4,$$

and, thus,

$$E_4E_3E_2E_1A = E_4(E_3E_2E_1A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 3 & 3 \\ 0 & -6 & -10 & -10 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & -6 & -10 & -10 \end{pmatrix}.$$

As illustrated in the example, above, performing a sequence of row operations to a matrix is equivalent to multiplying on the left by a sequence of elementary matrices. In particular, if $\widetilde{A}$ is the reduced row echelon form of A, then there are elementary matrices $E_1, \ldots, E_\ell$ such that

$$\widetilde{A} = E_\ell \cdots E_2E_1A.$$

Determinant of the transpose. If A is an $m \times n$ matrix, recall that its *transpose* is the matrix A^t defined by

$$(A^t)_{ij} := A_{ji}.$$

Thus, the rows of A^t are the columns of A.

Example If

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 5 & 2 \\ 1 & 3 \end{pmatrix},$$

then

$$A^t = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \quad \text{and} \quad B^t = \begin{pmatrix} 5 & 1 \\ 2 & 3 \end{pmatrix}.$$

Our goal now is to prove the non-obvious fact that for an $n \times n$ matrix A,

$$\det(A) = \det(A^t).$$

Example. We have seen that

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Note that we also have

$$\det \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^t \right) = \det \begin{pmatrix} a & c \\ b & d \end{pmatrix} = ad - bc.$$

Recall that we can compute the determinant of A by performing row operations and keeping track of swaps and scalings of rows. Once we have shown that $\det(A) = \det(A^t)$, it follows that, in order compute the determinant of A, we may also use column operations (again keeping track of swaps and scalings). That's because row operations applied to A^t are the same as column operations applied to A.

To prove this fact about determinants of transposes, we need the following theorem, proposition, and lemma:

Theorem. Let A and B be $n \times n$ matrices. Then

$$\det(AB) = \det(A)\det(B).$$

Proof. Upcoming homework. □

Proposition. Let A and B be $n \times n$ matrices. Then

(a) $(AB)^t = B^t A^t$.

(b) If A is invertible, then $(A^t)^{-1} = (A^{-1})^t$.

Proof. Upcoming homework. □

Lemma. Let E be an elementary matrix. Then $\det(E) = \det(E^t) \neq 0$.

Proof. There are three cases to consider[1]:

(a) Suppose E is formed from I_n by swapping rows i and j. In this case, E^t is also formed from I_n by swapping rows i and j. Thus, $E = E^t$, and $\det(E^t) = \det(E) = -1$.

[1] The reader is strongly encouraged to create examples of each of these three cases.

(b) Suppose E is formed from I_n by scaling row i by $\lambda \neq 0$. In this case, E^t is also formed from I_n by scaling row i by λ. So in this case, $\det(E^t) = \det(E) = \lambda$.

(c) Suppose E is formed from I_n by adding λr_i to r_j for rows $r_i \neq r_j$. Then E^t is formed from I_n by adding λr_j to r_i. So in this case, $\det(E^t) = \det(E) = \det(I_n) = 1$.

□

We can now prove our main result:

Theorem. Let A be an $n \times n$ matrix. Then $\det(A) = \det(A^t)$.

Proof. Let $\widetilde{A}$ be the reduced echelon form for A. Then $\widetilde{A} = I_n$ if and only if $\operatorname{rank} A = n$. So if $\widetilde{A} \neq I_n$, we have $\operatorname{rank}(A) < 0$, which means that $\det(A) = 0$. Since row rank and column rank are equal, we would then have $\operatorname{rank}(A^t) = \operatorname{rank}(A) < n$, which means $\det(A^t) = 0$, too. So the theorem holds in that case.

Now consider that case where $\widetilde{A} = I_n$. Thus, by applying row operations to A, we arrive at the identity matrix. Choose elementary matrices E_i such that

$$E_\ell \cdots E_2 E_1 A = I_n. \tag{1}$$

Taking determinants and using the fact that determinants preserve products yields:

$$\det(E_\ell) \cdots \det(E_2) \det(E_1) \det(A) = 1. \tag{2}$$

Taking transposes in equation (1) gives

$$A^t E_1^t \cdots E_\ell^t = I_n^t = I_n.$$

Take determinants and use the fact from the Lemma that $\det(E) = \det(E^t)$ if E is an elementary matrix to get

$$\det(A^t) \det(E_1) \cdots \det(E_\ell) = 1. \tag{3}$$

The result follows by equating (2) and (3) and using the fact that the determinant of an elementary matrix is nonzero (so that we may cancel). □

Corollary. The determinant is a multilinear, alternating, normalized function of the *columns* of a square matrix.

For instance, the above corollary says the in order to compute the determinant, one may use column operations to simplify the calculation, just as we used row operations. Further, one may mix row and column operations, when convenient, as in the following example.

Example. Let

$$M = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -2 & 3 & 0 & 1 \\ -2 & 1 & 4 & -1 \\ -5 & 1 & 1 & 5 \end{pmatrix}.$$

To compute the determinant of M, add the first row to each other row to get

$$M' = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & 0 & 3 & -2 \\ -4 & 0 & 0 & 4 \end{pmatrix},$$

then, in M', add columns 2, 3, and 4 to column 1 to get

$$M'' = \begin{pmatrix} -2 & -1 & -1 & -1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

The types of row and column operations we used to get from M to M'' have no effect on the determinant, and thus,

$$\det(M) = \det(M') = \det(M'') = -2 \cdot 2 \cdot 3 \cdot 4 = -48.$$

Remark. Start with I_n and perform a sequence of row operations to get a matrix $B \in M_{n\times n}(F)$. Then multiplying $A \in M_{n\times k}(F)$ on the *left* by B gives the matrix $BA \in M_{n\times k}(F)$ formed from A by performing the same sequence of row operations. Similarly, one can start with I_n, perform column operations to produce a matrix $C \in M_{n\times n}(F)$, then multiplying $D \in M_{k\times n}(F)$ on the *right* by C give the matrix $DC \in M_{k\times n}(F)$ obtained by performing the column operations on D.