Math 201 lecture for Friday, Week 8

Existence and uniqueness of the determinant

Laplace expansion of the determinant. Let A be an $n \times n$ matrix. For each $i, j \in \{1, 2, \ldots, n\}$, define A^{ij} to be the matrix formed by removing the i-th row and j-th column from A. Fix $k \in \{1, 2, \ldots, n\}$. Then

$$\det(A) = \sum_{j=1}^{n} (-1)^{k+j} A_{kj} \det(A^{kj}).$$

This expresses $\det(A)$ in terms of an alternating sum of determinants of $(n-1) \times (n-1)$ matrices. We call this *expanding* $\det(A)$ *along the k-th row*. Applying the formula recursively leads to a complete evaluation of $\det(A)$. Since, $\det(A) = \det(A^t)$, you can also calculate the determinant by recursively expanding along *columns*.

Example. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Let's calculate the determinant by expanding along the first row:

$$\det(A) = 1 \cdot \det \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} - 2 \cdot \det \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} + 3 \cdot \det \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= (-1) - 2(1) + 3(2) = 3.$$

To check, let's expand along the second row, instead, noting the signs:

$$\det(A) = -2 \cdot \det \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} 0 \cdot \det \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} - 1 \cdot \det \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

$$= -2(-1) + 0(-2) - 1(-1) = 3.$$

Finally, let's expand along the third *column*:

$$\det(A) = 3 \cdot \det \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} - 1 \cdot \det \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} + 1 \cdot \det \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$$

$$= 3(2) - 1(-1) + 1(-4) = 3.$$

Note: if your matrix has a particular row or column with a lot of 0s in it, you might want to expand along that row or column since a lot of the terms will be 0. For example, to compute

$$\det \begin{pmatrix} 1 & 3 & 0 \\ 3 & 2 & 3 \\ 1 & 4 & 0 \end{pmatrix},$$

expand along the third column:

$$0\,(\text{blah}) - 3 \det \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix} + 0\,(\text{blah}) = -3(1) = -3.$$

The "blah"s are there instead of explicit determinants since they are being multiplied by 0. Their exact values don't matter, so we don't need to waste time calculating them. We will not prove the formula for the Laplace expansion. It is very similar to that for the permutation expansion.

Existence and uniqueness of the determinant. Recall the definition that started our discussion of the determinant:

Definition. The *determinant* is a multilinear, alternating function $\det\colon M_{n\times n}(F) \to F$ of the rows of square matrix, normalized so that its value on the identity matrix is 1.

The definition says "the determinant", but for all we knew, there could be several different functions $M_{n\times n}(F) \to F$ all satisfying the criteria of being multilinear, alternating, and normalized. Or, it is possible there are no functions that satisfy the criteria? So the definition requires us to prove that, in fact, there exists exactly one determinant function (for each n).

Just after defining the determinant, we showed that if $d\colon M_{n\times n}(F) \to F$ is any multilinear, alternating, normalized function, then a choice of a row reduction for $A \in M_{n\times n}(F)$ determines the value of $d(A)$. The subtlety here is that, there are many different sequence of row operations that would produced the row echelon form for A. Do each of these produce the same value for $d(A)$ (in other words, is d *well-defined*)? We never proved that they would.

So let's begin again and consider the particular function $d\colon M_{n\times n}(F) \to F$ defined recursively as the Laplace expansion of a matrix along its first row:

$$d(A) := \sum_{j=1}^{n} (-1)^{1+j} A_{1j}\, d(A^{1j}) \tag{1}$$

if $n \geq 1$, and by $d(A) = a$ if $A = [a]$ is a 1×1 matrix. This function d is well-defined—there are no choices to be made is in calculation.

Exercise. Prove that d is multilinear, alternating, and normalized (i.e., its value at I_n is 1).

Thus, we see there exists at least one determinant function.

Having defined d by (1), now note that in addition to calculating d using the given recursive formula, since d is multilinear, alternating, and normalize, its value can be determined via row reductions, just as before. What's new now is that we see that no matter which choices are made in the row reduction, we must get the value determined by (1).

In sum, we have shown that a multilinear, alternating, normalized function exists and is unique. Its value is completely determined by choosing any sequence of row operations reducing a matrix to its row echelon form, and the choice of the sequence of row operations does not matter. So far, we have three different methods for calculating the determinant: using row operations, summing over permutations, and via Laplace expansion along any row or column.

BONUS CONTENT

Generalized Laplace expansion. Let $A \in M_{n\times n}(F)$, and fix a subset of k rows $r_{i_1}, \ldots, r_{i_k}$ of A where $1 \leq k \leq n$. Let $I = \{i_1, \ldots, i_k\}$ be the indices of these rows. For any subset $J \subseteq \{j_1, \ldots, j_k\}$, define $|J| := j_1 + \cdots + j_k$, and define

$$\begin{aligned} A^{IJ} = {} & \text{the } k \times k \text{ submatrix of } A \text{ formed by the intersection of rows} \\ & \text{indexed by } I \text{ and the columns indexed by } J \end{aligned}$$

$$\bar{A}^{IJ} = \text{the } (n-k)\times(n-k) \text{ submatrix of } A \text{ formed by the intersection of rows indexed by } \{1,\dots,n\}\setminus I \text{ and the columns indexed by } \{1,\dots,n\}\setminus J.$$

Then

$$\det(A) = \sum_J (-1)^{|I|+|J|}\det(A^{IJ})\det(\bar{A}^{IJ})$$

where the sum is over all k-element subsets J of $\{1,\dots,n\}$.[1]

Example. The case where $k=1$ is the ordinary Laplace expansion formula.

Example. Let

$$A = \begin{pmatrix} 1 & 7 & 0 & 5 \\ 2 & 2 & 2 & 2 \\ 5 & 1 & 4 & 6 \\ 0 & 6 & 7 & 3 \end{pmatrix}.$$

We will compute $\det(A)$ using the generalize Laplace expansion along the first two rows of A. So, using the notation from above, $I=\{1,2\}\subset\{1,2,3,4\}$. There are six choices for a pair of columns:

$$J \in (\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}).$$

The term in the expansion corresponding to $J=\{1,3\}$ would be

$$\begin{aligned}(-1)^{|I|+|J|}\det(A^{IJ})\det(\bar{A}^{IJ}) &= (-1)^{(1+2)+(1+3)}\det\begin{pmatrix}1 & 0\\ 2 & 2\end{pmatrix}\det\begin{pmatrix}1 & 6\\ 6 & 3\end{pmatrix}\\ &= (-1)(1\cdot 2-0\cdot 2)(1\cdot 3-6\cdot 6) = 66.\end{aligned}$$

The entire expansion is

$$\begin{aligned}\det(A) &= \sum_J (-1)^{|I|+|J|}\det(A^{IJ})\det(\bar{A}^{IJ})\\
&= \sum_J (-1)^{3+|J|}\det(A^{\{1,2\}J})\det(\bar{A}^{\{1,2\}J})\\
&= (-1)^{3+(1+2)}\det(A^{\{1,2\}\{1,2\}})\det(\bar{A}^{\{3,4\}\{3,4\}}) + (-1)^{3+(1+3)}\det(A^{\{1,2\}\{1,3\}})\det(\bar{A}^{\{3,4\}\{2,4\}})\\
&\quad + (-1)^{3+(1+4)}\det(A^{\{1,2\}\{1,4\}})\det(\bar{A}^{\{3,4\}\{2,3\}}) + (-1)^{3+(2+3)}\det(A^{\{1,2\}\{2,3\}})\det(\bar{A}^{\{3,4\}\{1,4\}})\\
&\quad + (-1)^{3+(2+4)}\det(A^{\{1,2\}\{2,4\}})\det(\bar{A}^{\{3,4\}\{1,3\}}) + (-1)^{3+(3+4)}\det(A^{\{1,2\}\{3,4\}})\det(\bar{A}^{\{3,4\}\{1,2\}})\\
&= \det(A^{\{1,2\}\{1,2\}})\det(\bar{A}^{\{3,4\}\{3,4\}}) - \det(A^{\{1,2\}\{1,3\}})\det(\bar{A}^{\{3,4\}\{2,4\}})\\
&\quad + \det(A^{\{1,2\}\{1,4\}})\det(\bar{A}^{\{3,4\}\{2,3\}}) + \det(A^{\{1,2\}\{2,3\}})\det(\bar{A}^{\{3,4\}\{1,4\}})\\
&\quad - \det(A^{\{1,2\}\{2,4\}})\det(\bar{A}^{\{3,4\}\{1,3\}}) + \det(A^{\{1,2\}\{3,4\}})\det(\bar{A}^{\{3,4\}\{1,2\}})\end{aligned}$$

[1] Since $\det(A)=\det(A^t)$, there is a similar formula for expansion along a fixed set of k columns.

Continuing the calculation:

$$A = \begin{pmatrix} 1 & 7 & 0 & 5 \\ 2 & 2 & 2 & 2 \\ 5 & 1 & 4 & 6 \\ 0 & 6 & 7 & 3 \end{pmatrix}$$

$$\begin{aligned}
\det(A) &= \det(A^{\{1,2\}\{1,2\}})\det(\bar{A}^{\{3,4\}\{3,4\}}) - \det(A^{\{1,2\}\{1,3\}})\det(\bar{A}^{\{3,4\}\{2,4\}}) \\
&\quad + \det(A^{\{1,2\}\{1,4\}})\det(\bar{A}^{\{3,4\}\{2,3\}}) + \det(A^{\{1,2\}\{2,3\}})\det(\bar{A}^{\{3,4\}\{1,4\}}) \\
&\quad - \det(A^{\{1,2\}\{2,4\}})\det(\bar{A}^{\{3,4\}\{1,3\}}) + \det(A^{\{1,2\}\{3,4\}})\det(\bar{A}^{\{3,4\}\{1,2\}}) \\
&= \det\begin{pmatrix} 1 & 7 \\ 2 & 2 \end{pmatrix}\det\begin{pmatrix} 4 & 6 \\ 7 & 3 \end{pmatrix} - \det\begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}\det\begin{pmatrix} 1 & 6 \\ 6 & 3 \end{pmatrix} \\
&\quad + \det\begin{pmatrix} 1 & 5 \\ 2 & 2 \end{pmatrix}\det\begin{pmatrix} 1 & 4 \\ 6 & 7 \end{pmatrix} + \det\begin{pmatrix} 7 & 0 \\ 2 & 2 \end{pmatrix}\det\begin{pmatrix} 5 & 6 \\ 0 & 3 \end{pmatrix} \\
&\quad - \det\begin{pmatrix} 7 & 5 \\ 2 & 2 \end{pmatrix}\det\begin{pmatrix} 5 & 4 \\ 0 & 7 \end{pmatrix} + \det\begin{pmatrix} 0 & 5 \\ 2 & 2 \end{pmatrix}\det\begin{pmatrix} 5 & 1 \\ 0 & 6 \end{pmatrix} \\
&= (-12)(-30) - (2)(-33) + (-8)(-17) + (14)(15) - (4)(35) + (-10)(30) \\
&= 332.
\end{aligned}$$

Example. Let

$$A = \begin{pmatrix} 3 & 2 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 0 \\ 0 & 1 & 7 & 0 & 0 \\ 1 & 2 & 1 & 4 & 7 \\ 3 & 4 & 2 & 9 & 3 \end{pmatrix}.$$

The generalized Laplace expansion along the first three rows has only two nonzero terms, yielding

$$\det(A) = \det\begin{pmatrix} 3 & 2 & 1 \\ 1 & 2 & 4 \\ 0 & 1 & 7 \end{pmatrix}\det\begin{pmatrix} 4 & 7 \\ 9 & 3 \end{pmatrix}.$$