Math 201 lecture for Wednesday, Week 7

Change of basis

Let V be a vector space with ordered basis $\alpha = \langle v_1, \ldots, v_n \rangle$. Recall the coordinate mapping

$$\phi_\alpha \colon V \xrightarrow{\sim} F^n$$
$$v = a_1 v_1 + \cdots + a_n v_n \mapsto (a_1, \ldots, a_n).$$

In particular, we have $\phi_\alpha(v_j) = e_j$. Consider the special case where $V = F^n$ so that

$$\phi_\alpha \colon F^n \to F^n,$$

and each v_j is an element of F^n. Since ϕ_α is now a mapping between tuples, is represented by the $n \times n$ matrix M with the property that $\phi_\alpha(v) = Mv$ for all $v \in F^n$. The j-th column of M is $\phi_\alpha(e_j)$ for $j = 1, \ldots, n$.

Proposition. With notation as above, let P be the $n \times n$ matrix whose j-th column is v_j for $j = 1, \ldots, n$. Then $M = P^{-1}$.

Proof. If X is any matrix, then Xe_j is the j-th column of X. So in our case, $Pe_j = v_j$. Since the v_j form a basis, the columns of P are linearly independent. So P has rank n and is, thus, invertible. We have

$$Pe_j = v_j \quad \Rightarrow \quad P^{-1}Pe_j = P^{-1}v_j \quad \Rightarrow \quad e_j = P^{-1}v_j.$$

Therefore, $P^{-1}v_j = e_j = \phi_\alpha(v_j)$ for all j. Since the v_j form a basis for F^n, it follows that $P^{-1}v = \phi_\alpha(v)$ for all $v \in F^n$. So P^{-1} is the matrix representing ϕ_α, which means that $P^{-1} = M$. $\square$

Next, consider a linear function

$$L_A \colon F^n \to F^m$$

given by the $m \times n$ matrix A, i.e., $L_A(v) = Av$ for all $v \in F^n$. Let $\alpha = \langle v_1, \ldots, v_n \rangle$ and $\beta = \langle w_1, \ldots, w_m \rangle$ be ordered bases for F^n and F^m respectively. What is the matrix representing L_A with respect to these new bases? We have the diagram

$$\begin{tikzcd} F^n \arrow[r, "L_A"] \arrow[d, "\phi_\alpha"', "\wr"] & F^m \arrow[d, "\phi_\beta", "\wr"'] \\ F^n \arrow[r, "{[L_A]_\alpha^\beta}"] & F^m. \end{tikzcd}$$

For ease of notation, let $B := [L_A]_\alpha^\beta$. Our main goal today is to give a formula for calculating B. We already know how to find the matrices representing the vertical coordinate mappings: let P and Q be the matrices whose columns are the elements of α and β, respectively, in order. Our diagram becomes

$$
\begin{array}{ccc}
F^n & \xrightarrow{A} & F^m \\
{\scriptstyle P^{-1}}\downarrow\wr & & \wr\downarrow{\scriptstyle Q^{-1}} \\
F^n & \xrightarrow{B} & F^m.
\end{array}
$$

Therefore, $B = Q^{-1}AP$. We summarize our result:

Proposition. Let $A \in M_{m\times n}(F)$, and consider the linear mapping $L_A\colon F^n \to F^m$ determined by A, i.e., $L(v) = Av$ for each $v \in F^n$. Let $\alpha = \langle v_1, \ldots, v_n\rangle$ and $\beta = \langle w_1, \ldots, w_m\rangle$ be ordered bases for F^n and F^m, respectively. Let P be the $n \times n$ matrix with j-th column v_j for $j = 1, \ldots, n$, and let Q be the $m \times m$ matrix with j-th column w_j for $j = 1, \ldots, m$. Then the matrix B representing L_A with respect to the bases α and β is

$$B = Q^{-1}AP,$$

and we have the commutative diagram

$$
\begin{array}{ccc}
F^n & \xrightarrow{A} & F^m \\
{\scriptstyle P^{-1}}\downarrow\wr & & \wr\downarrow{\scriptstyle Q^{-1}} \\
F^n & \xrightarrow{B} & F^m.
\end{array}
$$

Example. Let $\mathbb{Q}$ be the field of rational numbers, and consider the linear function

$$
\begin{aligned}
f\colon \mathbb{Q}^3 &\to \mathbb{Q}^2 \\
(x, y, z) &\mapsto (x + 3y + 2z, 2y + z),
\end{aligned}
$$

with corresponding matrix

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \end{pmatrix}.$$

Choose the following bases for the domain and codomain:

$$
\begin{aligned}
\mathbb{Q}^3: &\quad \alpha = \langle (1, 0, 0), (1, 1, 0), (1, 1, 1)\rangle \\
\mathbb{Q}^2: &\quad \beta = \langle (0, 1), (1, 1)\rangle.
\end{aligned}
$$

To find the matrix representing f with respect to these new bases, create matrices whose columns are the basis vectors:

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Calculate the inverse of Q:

$$\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow{r_1 \to r_1 - r_2} \left(\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right).$$

The matrix representing f with respect to the bases α and β is then:

$$B = Q^{-1}AP = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 & -3 \\ 1 & 4 & 6 \end{pmatrix}.$$

This agrees with the fact that

$$\begin{aligned} f(1,0,0) &= (1,0) = -1(0,1) + 1(1,1) \\ f(1,1,0) &= (4,2) = -2(0,1) + 4(1,1) \\ f(1,1,1) &= (6,3) = -3(0,1) + 6(1,1). \end{aligned}$$

An important special case. The special case of the Proposition that arises most frequently in practice is where $m = n$ and $\alpha = \beta$. In other, words, we start with a mapping $L_A\colon F^n \to F^n$ represented by the matrix A, and we choose the same new basis $\alpha = \langle v_1, \ldots, v_n \rangle$ for F^n for both the domain and codomain. We are then interested in the matrix representing L_A with respect to this new basis α. In that case, let P be the matrix whose columns are $v_1, \ldots, v_n$, and we get the commutative diagram

$$\begin{tikzcd} F^n \arrow[r, "A"] \arrow[d, "P^{-1}"', "\wr"] & F^n \arrow[d, "P^{-1}", "\wr"'] \\ F^n \arrow[r, "B"] & F^m \end{tikzcd}$$

and the matrix we are looking for is

$$B = P^{-1}AP.$$

We say B is formed by *conjugating* A.

Exercise. Say $A, B \in M_{n\times n}(F)$ are *similar* and write $A \sim B$ if there exists an invertible matrix $P \in M_{n\times n}(F)$ such that $P^{-1}AP = B$. Prove that $\sim$ is an equivalence relation. What does an equivalence class represent?

Example. Consider the real matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

What matrix represents the linear function $L_A\colon F^3 \to F^3$ with respect to the ordered basis $\alpha = \langle (1,1,1), (1,0,-1), (0,1,-1) \rangle$?

Solution. Use the vectors of α as columns to define the matrix

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}.$$

Compute the inverse of P using our algorithm (omitted):

$$P^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix}.$$

Then the matrix representing L_A with respect to the ordered basis α is

$$B = P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Notice that the matrix representing L_A in the example becomes the much simpler diagonal matrix after a change of basis. We can then apply an important trick to compute A^k for all integers k. First note that

$$A^2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}, \quad A^4 = \begin{pmatrix} 6 & 5 & 5 \\ 5 & 6 & 5 \\ 5 & 5 & 6 \end{pmatrix}, \quad A^5 = \begin{pmatrix} 10 & 11 & 11 \\ 11 & 10 & 11 \\ 11 & 11 & 10 \end{pmatrix}.$$

What happens in general? Here is the trick that can be applied here:

$$\begin{aligned} B^k &= (P^{-1}AP)^k \\ &= \underbrace{(P^{-1}AP)(P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP)(P^{-1}AP)}_{k \text{ times}} \\ &= P^{-1}A(PP^{-1})A(PP^{-1})A(PP^{-1})\cdots(PP^{-1})A(PP^{-1})AP \\ &= P^{-1}A^kP. \end{aligned}$$

Since $B^k = P^{-1}A^kP$, we can solve for A^k by multiply both sides of the equality on the left by P and on the right by P^{-1} to get

$$\begin{aligned}
A^k = PB^kP^{-1} &= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}^k \left(\frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix} \right) \\
&= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 2^k & 0 & 0 \\ 0 & (-1)^k & 0 \\ 0 & 0 & (-1)^k \end{pmatrix} \left(\frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix} \right) \\
&= \frac{1}{3} \begin{pmatrix} 2^k + 2\,(-1)^k & 2^k - (-1)^k & 2^k - (-1)^k \\ 2^k - (-1)^k & 2^k + 2\,(-1)^k & 2^k - (-1)^k \\ 2^k - (-1)^k & 2^k - (-1)^k & 2^k + 2\,(-1)^k \end{pmatrix} \\
&= \frac{1}{3} \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}
\end{aligned}$$

where for $k = 1, 2, 3, \ldots,$

$$a = \begin{cases} 2^k + 2 & \text{if } k \text{ is even} \\ 2^k - 2 & \text{if } k \text{ is odd} \end{cases} \quad \text{and} \quad b = \begin{cases} 2^k - 1 & \text{if } k \text{ is even} \\ 2^k + 1 & \text{if } k \text{ is odd.} \end{cases}$$

Exercise. Show that $2^k \pm 2$ and $2^k \pm 1$ are divisible by 3 for $k = 1, 2, \ldots$