Math 201 lecture for Friday, Week 7

Determinants

Definition. The *determinant* is a multilinear, alternating function of the rows of square matrix, normalized so that its value on the identity matrix is 1.

To explain this terminology, start with the fact that the determinant is a function of the form

$$\det\colon M_{n\times n}(F) \to F.$$

Given a square matrix $A \in M_{n\times n}(F)$ with rows $r_1, \dots, r_n \in F^n$, we write $\det(A) = \det(r_1, \dots, r_n)$, i.e., we consider the determinant as a function of the rows of A. The determinant function has the following properties:

(a) *Multilinear.* The determinant is a linear function with respect to each row. Thus, if $r_1, \dots, r_n$ are the row vectors of A (elements of F^n), r_i' is another row vector, and $\lambda \in F$, then

$$\begin{aligned}\det(r_1, \dots, r_{i-1}, \lambda\, r_i + r_i', r_{i+1}, \dots, r_n) = \lambda\ &\det(r_1, \dots, r_{i-1}, r_i, r_{i+1}, \dots, r_n) \\ &+ \det(r_1, \dots, r_{i-1}, r_i', r_{i+1}, \dots, r_n).\end{aligned}$$

The above expresses the fact that, in particular, the determinant is linear with respect to the i-th row.

(b) *Alternating.* The determinant is zero if two of its arguments are equal:

$$\det(r_1, \dots, r_n) = 0$$

if $r_i = r_j$ for some $i \neq j$.

(c) *Normalized.* $\det(I_n) = \det(e_1, \dots, e_n) = 1$.

We will prove the following theorem later:

Theorem. For each $n \geq 0$, there exists a unique determinant function.

For now we will accept this theorem on faith and explore some of the consequences. The following proposition shows that we can compute the determinant through row reduction.

Proposition 1. (Behavior of the determinant with respect to row operations.) Let $A, B \in M_{n\times n}(F)$.

(a) If B is obtained from A by swapping two rows, then $\det(B) = -\det(A)$.

(b) If B is obtained from A by scaling a row by a scalar λ, then $\det(B) = \lambda \det(A)$.

(c) If B is obtained from A by adding a scalar multiple of one row to another row, then $\det(B) = \det(A)$.

Proof. For part (a), let $r_1, \ldots, r_n \in F^n$ be the rows of A. For ease of notation, we will assume that B is obtained from A by swapping the first two rows. The argument we present clearly generalizes to the case of swapping arbitrary rows. Replace the first two rows of A with $r_1 + r_2$ to obtain a matrix whose determinant is 0 by the alternating property:

$$0 = \det(r_1 + r_2, r_1 + r_2, r_3, \ldots, r_n).$$

Expand my multilinearity to get:

$$\begin{aligned} 0 &= \det(r_1 + r_2, r_1 + r_2, r_3, \ldots, r_n) \\ &= \det(r_1, r_1 + r_2, r_3, \ldots, r_n) + \det(r_2, r_1 + r_2, r_3, \ldots, r_n) \\ &= \det(r_1, r_1, r_3, \ldots, r_n) + \det(r_1, r_2, r_3, \ldots, r_n) \\ &\qquad\qquad + \det(r_2, r_1, r_3, \ldots, r_n) + \det(r_2, r_2, r_3, \ldots, r_n) \\ &= 0 + \det(A) + \det(B) + 0. \end{aligned}$$

It follows that $\det(B) = -\det(A)$.

Part (a) follows immediately from the fact that the determinant is linear with respect to each row:

$$\det(r_1, \ldots, r_{i-1}, \lambda r_i, r_{i+1}, \ldots, r_n) = \lambda \det(r_1, \ldots, r_{i-1}, r_i, r_{i+1}, \ldots, r_n).$$

For Part (c), we use multilinearity and the alternating property. For ease of notation, we'll consider the case where B is obtained from A by adding a multiple of row 1 to row 2:

$$\begin{aligned} \det(B) &= \det(r_1, \lambda r_1 + r_2, r_3, \ldots, r_n) \\ &= \lambda \det(r_1, r_1, r_3, \ldots, r_n) + \det(r_1, r_2, r_3, \ldots, r_n) \\ &= 0 + \det(r_1, r_2, r_3, \ldots, r_n) \\ &= \det(A). \end{aligned}$$

□

Corollary. Let $A \in M_{n\times n}(F)$, and let E be the reduced row echelon form of A. Then there exists a non-zero $k \in F$ such that $\det(A) = k \det(E)$.

Proof. The proof is immediate from Proposition 1. □

Example 1. Here we compute the determinant of a 2×2 matrix using the fact that the determinant is a multilinear alternating mapping with value 1 on the identity matrix.

$$
\begin{aligned}
\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \det((a, b), (c, d)) \\
&= \det(a\, e_1 + b\, e_2, c\, e_1 + d\, e_2) \\
&= a \det(e_1, c\, e_1 + d\, e_2) + b \det(e_2, c\, e_1 + d\, e_2) \\
&= ac \det(e_1, e_1) + ad \det(e_1, e_2) + bc \det(e_2, e_1) + bd \det(e_2, e_2) \\
&= 0 + ad \det(e_1, e_2) + bc \det(e_2, e_1) + 0 \\
&= ad \det(e_1, e_2) - bc \det(e_1, e_2) \\
&= ad \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - bc \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= ad \cdot 1 - bc \cdot 1 = ad - bc.
\end{aligned}
$$

Example 2. Here is an example of using row reduction to compute the determinant of a matrix. Let

$$
A = \begin{pmatrix} 1 & 2 & -2 \\ 9 & 4 & 0 \\ 2 & 2 & 4 \end{pmatrix}
$$

Using Proposition 1, we see that

$$
\begin{aligned}
\det(A) &= \det\begin{pmatrix} 1 & 2 & -2 \\ 9 & 4 & 0 \\ 2 & 2 & 4 \end{pmatrix} \\[2ex]
&= \det\begin{pmatrix} 1 & 2 & -2 \\ 0 & -14 & 18 \\ 0 & -2 & 8 \end{pmatrix} \\[2ex]
&= -\det\begin{pmatrix} 1 & 2 & -2 \\ 0 & -2 & 8 \\ 0 & -14 & 18 \end{pmatrix} \\[2ex]
&= 2\det\begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & -4 \\ 0 & -14 & 18 \end{pmatrix} \\[2ex]
&= 2\det\begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & -4 \\ 0 & 0 & -38 \end{pmatrix} \\[2ex]
&= 2(-38)\det\begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix} \\[2ex]
&= 2(-38)\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\[2ex]
&= 2(-38) = -76.
\end{aligned}
$$

Example 3.

$$\det\begin{pmatrix} 4 & 2 & -3 & 8 \\ 0 & 5 & 1 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \end{pmatrix} = (4\cdot 5\cdot 2\cdot 3)\det\begin{pmatrix} 1 & 1/2 & -3/2 & 4 \\ 0 & 1 & 1/5 & 3/5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= (4\cdot 5\cdot 2\cdot 3)\det\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= (4\cdot 5\cdot 2\cdot 3)\cdot 1 = 120.$$

A matrix like that in the previous example, which has only zero entries below the diagonal, is called *upper-triangular*. So $A \in M_{n\times n}(F)$ is upper-triangular if $A_{ij} = 0$ whenever $i > j$.

Proposition 2. The determinant of an upper-triangular matrix is the product of its diagonal elements.

Proof. Let A be upper-triangular, and let E be its reduced echelon form. From Proposition 1, we know that $\det(A) = k\det(E)$ for some non-zero constant k. Imagine row-reducing an upper-triangular matrix, and you will see that E has a row of zeros if and only if A has some diagonal entry equal to zero. If E has a row of zeros, then $\det(E) = 0$. To see this, suppose the rows of E are $r_1, \ldots, r_n$ with $r_n = \vec{0}$. By multilinearity, we have:

$$\begin{aligned} \det(E) &= \det(r_1, \ldots, r_{n-1}, \vec{0}) \\ &= \det(r_1, \ldots, r_{n-1}, 0\cdot\vec{0}) \\ &= 0\cdot\det(r_1, \ldots, r_{n-1}, \vec{0}) \\ &= 0. \end{aligned}$$

So if A has a diagonal entry equal to 0, then $\det(E) = 0$, which implies $\det(A) = k\det(E) = 0$. So the result holds in this case.

Next, suppose that A has no diagonal entries equal to 0. Compute $\det(A)$ using multilinearity:

$$\det(A) = \det \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & a_{24} & \dots & a_{2n} \\ 0 & 0 & a_{33} & a_{34} & \dots & a_{3n} \\ 0 & 0 & 0 & a_{44} & \dots & a_{4n} \\ & & & & \ddots & \vdots \\ & & & & & a_{nn} \end{pmatrix}$$

$$= a_{11} \cdots a_{nn} \det \begin{pmatrix} 1 & a_{12}/a_{11} & a_{13}/a_{11} & a_{14}/a_{11} & \dots & a_{1n}/a_{11} \\ 0 & 1 & a_{23}/a_{22} & a_{24}/a_{22} & \dots & a_{2n}/a_{22} \\ 0 & 0 & 1 & a_{34}/a_{33} & \dots & a_{3n}/a_{33} \\ 0 & 0 & 0 & 1 & \dots & a_{4n}/a_{44} \\ & & & & \ddots & \vdots \\ & & & & & 1 \end{pmatrix}$$

$$= a_{11} \cdots a_{nn} \det(I_n)$$
$$= a_{11} \cdots a_{nn}.$$

Above, it is clear that once we get to the case of all 1s on the diagonal, we can row-reduce the matrix to the identity by adding multiples of rows to other rows—operations that do not change the determinant. □

Proposition 3. Let $A \in M_{n \times n}(F)$. The following are equivalent:

(a) $\det(A) \neq 0$,
(b) $\operatorname{rank}(A) = n$,
(c) A is invertible, i.e., A has an inverse.

Proof. Given our algorithm for computing the inverse of a matrix, the equivalence of parts 2 and 3 is evident. To show that parts 1 and 2 are equivalent, recall that by Proposition 1, we have $\det(A) = k \det(E)$ where E is the reduced echelon form of A and k is a non-zero scalar. Thus, $\det(A) = 0$ if and only if $\det(E) = 0$. The rank of A is n if and only if $E = I_n$, in which case $\det(A) = k \neq 0$. The rank of A is strictly less than n if and only if E has a row of zeros. Since E is upper-triangular, Proposition 2 implies that E has a row of zeros if and only if $\det(E) = 0$. □

To come:

(a) Define the *transpose*, A^t of A by $A^t_{ij} := A_{ji}$. Then $\det A^t = \det A$, and thus, the determinant is also the unique multilinear, alternating, normalized function on the *columns* of a matrix.

(b) The determinant is multiplicative: $\det(AB) = \det(A)\det(B)$.

(c) The determinant may be calculated by "expanding" along any row or column.

(d) We have the following formula for the determinant

$$\det A = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) A_{1\sigma(1)} \cdots A_{n\sigma(n)}$$

where $\mathfrak{S}_n$ is the collection of all permutations of $(1, \ldots, n)$ and $\operatorname{sgn}(\sigma)$ is the *sign* of the permutation σ (i.e., 1 if the permutation is formed by an even number of flips and -1 if it is formed by an odd number of flips).

(e) Over the real numbers, the determinant gives the signed volume of the parallelepiped spanned by the rows (or by the columns) of the matrix.