Math 201 lecture for Wednesday, Week 5

Range and nullspace

Recall the definition of a linear function from last time: a function $f\colon V \to W$ between vectors spaces V and W over the (same) field F is a function $f\colon V \to W$ that preserves addition and scalar multiplication. In detail, this means that for all $u, v \in V$ and $\lambda \in F$,

$$f(u+v) = f(u) + f(v) \quad \text{and} \quad f(\lambda v) = \lambda f(v).$$

Definition/Proposition 1. Suppose $f\colon V \to W$ is linear and $U \subseteq V$ is a subspace of V. Then the *image of U under f* is the set

$$f(U) := \{f(u) : u \in U\} \subseteq W.$$

The image of U under f is a subspace of W.

Proof. Since U is a subspace of V, it follows that $0_V \in U$, and hence, $f(0_V) = 0_W \in f(U)$. Thus, $f(U)$ is nonempty. Next, let $x, y \in f(U)$, and let $\lambda \in F$. By definition of $f(U)$, there are vectors $u, v \in U$ such that $f(u) = x$ and $f(v) = y$. Then since f is linear, is preserves addition and scalar multiplication. Therefore,

$$\begin{aligned} x + \lambda y &= f(u) + \lambda f(v) \\ &= f(u) + f(\lambda v) \\ &= f(u + \lambda v). \end{aligned}$$

Since U is a subspace, it is closed under addition and scalar multiplication. Therefore, $u + \lambda v \in U$. It follows that $x + \lambda y = f(u + \lambda v) \in f(U)$, as required. $\square$

In particular, since V is a subspace of itself, its image under a linear function is a subspace of the codomain of the function.

Definition. The *image* or *range* of a linear function $f\colon V \to W$ is the subspace

$$\operatorname{im}(f) := \mathcal{R}(f) := f(V) := \{f(v) : v \in V\} \subseteq W.$$

The dimension of the image of f is the *rank* of f (provided it is finite-dimensional) and is denoted $\operatorname{rank}(f)$ or $\operatorname{rk}(f)$.

Example. Define a linear function $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ by letting $f(1,0) = (2,1,0)$ and $f(0,1) = (0,-1,1)$ and extending linearly. Thus, for all $x, y \in \mathbb{R}$,

$$\begin{aligned} f(x,y) &= f(x(1,0) + y(0,1)) \\ &= xf(1,0) + yf(0,1) \\ &= x(2,1,0) + y(0,-1,1). \end{aligned}$$

We have

$$\operatorname{im}(f) = \mathcal{R}(f) = \operatorname{Span}\{(2,1,0),(0,-1,1)\}.$$

Since $(2,1,0)$ and $(0,-1,1)$ are linearly independent and span the image, they are a basis for the image of f, and thus, $\operatorname{rank}(f) = 2$.

Remark. If $f\colon V \to W$ is a linear function, and $B = \{b_1, \ldots, b_n\}$ is a basis for V, then

$$\operatorname{im}(f) = \operatorname{Span}(f(B)) := \operatorname{Span}\{f(b_1), \ldots, f(b_n)\}$$

since $f(\sum_{i=1}^n \alpha_i b_i) = \sum_{i=1}^n \alpha_i f(b_i)$. However, $f(B)$ is not necessarily a basis for $\operatorname{im}(f)$.

Definition/Proposition 2. Let $f\colon V \to W$ be a linear mapping, and let U be a subspace of W. Then the *inverse image of* U *under* f is the set

$$f^{-1}(U) := \{v \in V : f(v) \in U\} \subseteq V.$$

The inverse image of U under f is a subspace of V.

Proof. Since U is a subspace of W, we know $0_W \in U$. Then, since $f(0_V) = 0_W$, it follows that $0_V \in f^{-1}(U)$. So $f^{-1}(U)$ is nonempty. Next, let $v, v' \in f^{-1}(U)$, and let $\lambda \in F$. It follows that $f(v) \in U$ and $f(v') \in U$. Since U is a subspace, it follows that $f(v) + \lambda f(v') \in U$. Since f is linear,

$$f(v + \lambda v') = f(v) + \lambda f(v') \in U.$$

It follows that $v + \lambda v' \in f^{-1}(U)$. □

Definition. Let $f\colon V \to W$ be a linear mapping. The *kernel* or *nullspace* of f, denoted $\ker f$ or $\mathcal{N}(f)$, respectively, is the inverse image of $\{0_W\}$:

$$\ker(f) := \mathcal{N}(f) := f^{-1}(\{0_W\}) := \{v \in V : f(v) = 0\}.$$

It is a subspace of V (by Proposition 2). The dimension of the kernel is called the *nullity* of f (provided it is finite-dimensional) and is denoted $\operatorname{nullity}(f)$.

Example. Consider the linear mapping

$$\begin{aligned} f\colon \mathbb{R}^2 &\to \mathbb{R}^3 \\ (x,y) &\mapsto (2x, x-y, y). \end{aligned}$$

To find the kernel of f, we look for vectors (x,y) such that

$$f(x,y) = (2x, x-y, y) = (0,0,0).$$

Comparing vector components, we see that $x = y = 0$ is the only possibility. Therefore,

$$\ker(f) = \{(0,0)\},$$

and nullity$(f) = 0$.

Example. Let $\mathbb{R}[x]_{\leq 2}$ denote polynomials in x of degree at most two and with real coefficients. Consider the linear mapping

$$\begin{aligned} f\colon \mathbb{R}[x]_{\leq 2} &\longrightarrow \mathbb{R}^2 \\ a + bx + cx^2 &\mapsto (a+b, a+c). \end{aligned}$$

To find the kernel of f, we need to find a, b, c such that $f(a + bx + cx^2) = (0, 0)$. This amounts to solving the system of equations

$$\begin{aligned} a + b &= 0 \\ a + c &= 0. \end{aligned}$$

Apply our algorithm:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right). \tag{$\star$}$$

Solving for the pivot variables, we get

$$\begin{aligned} a &= -c \\ b &= c. \end{aligned}$$

Therefore,

$$\ker(f) = \left\{-c + cx + cx^2 : c \in \mathbb{R}\right\} = \operatorname{Span}\left\{-1 + x + x^2\right\}.$$

Therefore, the nullity of f is $\dim(\ker(f) = 1$. A basis for $\mathbb{R}[x]_{\leq 2}$ is the set $\left\{1, x, x^2\right\}$, and the image of these vectors forms a basis for the image of f:

$$f(1) = (1,1), \quad f(x) = (1,0), \quad f(x^2) = (0,1).$$

So the image of f is the column space of the matrix for the linear system we solved to find the kernel (cf. Equation ($\star$)). Using our algorithm for finding the basis of the column space, we get the basis $\{(1,1),(1,0)\}$. Another basis is $\{(1,0),(0,1)\}$. Therefore, the rank of f is rank$(f) = 2$.

Our main goal next time will to prove the following:

Theorem. (Rank-nullity theorem) Let $f\colon V \to W$ be a linear mapping, and suppose that V is finite-dimensional. Then

$$\operatorname{rank}(f) + \operatorname{nullity}(f) = \dim V.$$

In other words, $\dim(\operatorname{im}(f)) + \dim(\ker(f)) = \dim V$.

Example. In the previous example, we found

$$\operatorname{rank}(f) + \operatorname{nullity}(f) = 2 + 1 = 3 = \dim \mathbb{R}[x]_{\leq 2}.$$