Math 201 lecture for Monday, Week 5

Linear transformations

Linear transformations. We have now defined the objects of study—vector spaces. Next, we need to consider the appropriate mappings between those objects—those that preserve the linear structure.

Definition. Let V and W be vector spaces over a field F. A *linear transformation* from V to W is a function

$$f\colon V \to W$$

satisfying, for all $v, v' \in V$ and $\lambda \in F$,

$$f(v + v') = f(v) + f(v') \quad \text{and} \quad f(\lambda v) = \lambda f(v).$$

Remarks. Using the notation from the definition:

- If $f(v+v') = f(v)+f(v')$, we say f *preserves addition.* Note that the addition on the left side is in V and the addition on the right side is in W. Thus, if $V \neq W$, they are two different operations (with the same name). Similarly, if $f(\lambda v) = \lambda f(v)$, we say f *preserves scalar multiplication.*

- One may combine the two conditions, above, for linearity into one: for f to be linear, we require
$$f(v + \lambda v') = f(v) + \lambda f(v')$$
for all $v, v' \in V$ and $\lambda \in F$.

- Synonyms for "linear transformation" are: "linear mapping" and "linear homomorphism", often with the word "linear" dropped when clear from context (and it will be since this is a course in linear algebra!).

- Our book restricts "linear transformation" to mean a linear transformation of the form $f\colon V \to V$, where the domain and codomain are equal. That is non-standard, and we won't use that terminology. Linear mappings from a vector space to itself are called linear *endomorphisms* or linear *self-mappings.*

Template for a proof that a mapping is linear. Consider the function

$$\begin{aligned} f\colon \mathbb{R}^3 &\to \mathbb{R}^2 \\ (x, y, z) &\mapsto (2x + 3y, x + y - 3z). \end{aligned}$$

Claim: f is linear.

Proof. Let $(x, y, z), (x', y', z') \in \mathbb{R}^3$ and $\lambda \in R$.

$$\begin{aligned}
f((x,y,z)+(x',y',z')) &= f(x+x', y+y', z+z') \\
&= (2(x+x')+3(y+y'), (x+x')+(y+y')-3(z+z')) \\
&= ((2x+3y)+(2x'+3y'), (x+y-3z)+(x'+y'-3z')) \\
&= (2x+3y, x+y-3z)+(2x'+3y', x'+y'-3z') \\
&= f(x,y,z)+f(x',y',z').
\end{aligned}$$

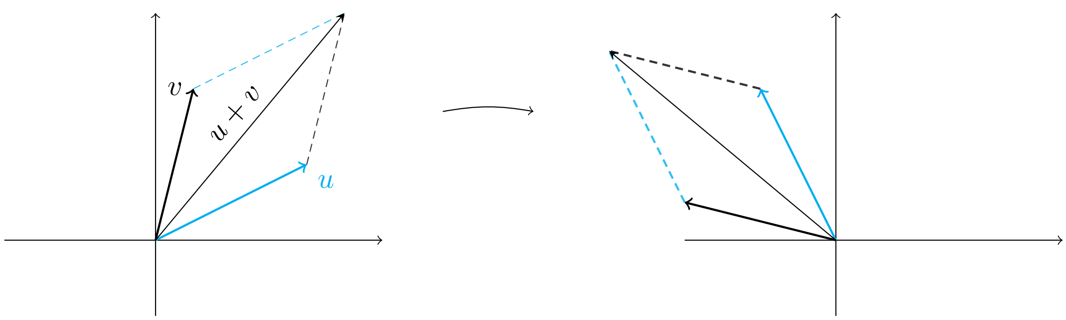
Thus, f preserves addition. Next,

$$\begin{aligned}
f(\lambda(x,y,z)) &= f(\lambda x, \lambda y, \lambda z) \\
&= (2(\lambda x)+3(\lambda y), (\lambda x+\lambda y-(3\lambda z)))) \\
&= (\lambda(2x+3y), \lambda(x+y-3z)) \\
&= \lambda(2x+3y, x+y-3z) \\
&= \lambda f(x,y,z).
\end{aligned}$$

Thus, f preserves scalar multiplication. $\square$

Note: People sometimes confuse proofs that subsets are subspaces with proofs that mappings are linear. To prove that $W \subseteq V$ is a subspace, we show that W is *closed under* addition and scalar multiplication by taking $u, v \in W$ and $\lambda \in F$ and showing $u + \lambda v \in W$. To prove $f\colon V \to W$ is linear, we show that f *preserves* addition and scalar multiplication. Be careful not to confuse the words "closed under" with "preserves".

Example. Rotation about the origin in the plane $\mathbb{R}^2$ is a linear transformation:



Exercise. Show that $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not linear.

Proof. We have $f(1+1) = f(2) = 4 \neq f(1)+f(1) = 1+1 = 2$. $\square$

The following proposition is often useful for showing a function is not linear.

Proposition 1. If $f\colon V \to W$ is linear, then $f(\vec{0}_V) = \vec{0}_W$.

PROOF. Since f is linear,

$$f(\vec{0}_V) = f(0 \cdot \vec{0}_V) = 0 \cdot f(\vec{0}_V) = \vec{0}_W.$$

Thus, for instance,

$$\begin{aligned} f \colon \mathbb{R}^2 &\to \mathbb{R} \\ (x, y) &\mapsto x + 2y + 5 \end{aligned}$$

is not linear since $f(0,0) = 5 \neq 0$.

Proposition 2. (A linear mapping is determined by its action on a basis.) Let V and W be vector spaces over F, and let B be a basis for V. For each $b \in B$, let $w_b \in W$. Then there exists a unique linear function $f \colon V \to W$ such that $f(b) = w_b$.

Proof. We define f as follows: Given $v \in V$, since B is a basis, we can write $v = \alpha_1 b_1 + \cdots + \alpha_k b_k$ for some $\alpha_i \in F$, $b_i \in B$, and $k \in \mathbb{Z}_{\geq 0}$. Define

$$f(v) := \alpha_1 f(b_1) + \ldots \alpha_k f(b_k) = \alpha_1 w_{b_1} + \cdots + \alpha_k w_{b_k}.$$

Since B is a basis, the expression for v as a linear combination of elements in B is unique. Hence, f is well-defined. Further, linearity of f forces us to define $f(v)$ as we have. To see that f is linear, let $v, w \in V$ and $\lambda \in \mathbb{R}$. Write v and w as linear combinations of the basis vectors:

$$\begin{aligned} v &= \alpha_1 b_1 + \cdots + \alpha_k b_k \\ w &= \beta_1 b_1 + \cdots + \beta_k b_k \end{aligned}$$

for some scalars α_i and β_i. It follows that

$$v + \lambda w = (\alpha_1 + \lambda\beta_1) b_1 + \cdots + (\alpha_k + \lambda\beta_k) b_k.$$

Using the definition of f, we see

$$\begin{aligned} f(v + \lambda w) &= (\alpha_1 + \lambda\beta_1) w_{b_1} + \cdots + (\alpha_k + \lambda\beta_k) w_{b_k} \\ &= (\alpha_1 w_{b_1} + \cdots + \alpha_k w_{b_k}) + \lambda(\beta_1 w_{b_1} + \cdots + \beta_k w_{b_k}) \\ &= f(v) + \lambda f(w). \end{aligned}$$

$\square$

Terminology. We say the function f as in Proposition 2 has been defined on B then *extended linearly* to all of V.

Example. Define a linear function $f \colon \mathbb{R}^2 \to M_{2\times 3}(\mathbb{R})$ by

$$f(1,0) = \begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 2 \end{pmatrix} \quad \text{and} \quad f(0,1) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}.$$

What is $f(2,-1)$?

Solution. In general, we have

$$\begin{aligned} f(x,y) &= f\left(x(1,0)+y(0,1)\right) \\ &= xf(1,0)+yf(0,1) \\ &= x\begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 2 \end{pmatrix} + y\begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} \\ &= \begin{pmatrix} x+2y & y & 2x \\ 3x & -x+3y & 2x+y \end{pmatrix}. \end{aligned}$$

In particular,

$$f(2,-1) = 2\begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 4 \\ 6 & -5 & 3 \end{pmatrix}.$$

Question. What goes wrong if we try to define a linear function by specifying its values on a non-basis? For instance, what happens if we try to define a linear function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ by specifying the values for the non-basis $\{(1,0),(2,0)\}$ as follows:

$$f(1,0) = (3,2) \quad \text{and} \quad f(2,0) = (1,1).$$

Note. Let V and W be vector spaces over F, and let X be a linearly subset of V. For each $x \in X$, let $w_x \in W$. Then there exists a linear function $f\colon V \to W$ such that $f(x) = w_x$ for all $x \in W$. To see this, let B be any completion of X to a basis for V, and apply Proposition 2. The map created this way is not unique: we are free to choose any values for elements of $B \setminus X$ (the value $\vec{0}$ might be a natural choice).

Here is something interesting that we will talk more about later:

Definition. Let V and W be vector spaces over F. The collection of all linear functions from V to W is denoted $\mathrm{Hom}(V,W)$ or $\mathcal{L}(V,W)$. It is a vector space over F under addition and scalar multiplication of functions: for linear $f,g\colon V \to W$,

$$\begin{aligned} f+\lambda g\colon V &\to W \\ v &\mapsto f(v)+\lambda g(v). \end{aligned}$$