Math 201 lecture for Wednesday, Week 4

Dimension II

Last time, we showed that if V is finite-dimensional, then all of its bases have the same number of elements. Then, by definition, the number of elements in any basis for V is the *dimension* of V.

Examples.

– $\dim F^n = n$ (for instance, $\{e_1, \dots, e_n\}$ is a basis).

– $\dim \mathcal{P}_d(F) = \dim F[x]_{\leq d} = d + 1$ (for instance, $\{1, x, \dots, x^d\}$ is a basis).

– $\dim\{(x, y, z) \in F^3 : x+y+z = 0\} = 2$ (for instance, $\{(1, 0, -1), (0, 1, -1)\}$ is a basis).

– $\dim_{\mathbb{R}} \mathbb{C} = 2$ (for instance, $\{1, i\}$ is a basis).

– $\dim_{\mathbb{C}} \mathbb{C} = 1$ (for instance, $\{1\}$ is a basis).

– $\dim\{\vec{0}\} = 0$ (the basis is $\emptyset$, which has 0 elements).

Corollary. Let V be a vector space of dimension n. Then

(a) If $S \subseteq V$ is linearly independent, then S has at most n elements.

(b) If $S \subseteq V$ is linearly independent, then S can be completed to a basis for V, i.e., there exists a basis containing S as a subset.

(c) If S has n elements, then S is linearly independent if and only if it spans V.

(d) If S spans V, then S has at least n elements.

(e) A basis is a minimal spanning set for V. (Here, "minimal" can mean the set has no strict subsets that also span V, or it can mean minimal in number of elements.)

(f) A basis is a maximal linearly independent subset of V. (Here, "maximal" can mean there is no strict superset that is also linearly independent, or it can mean maximal in number.)

Example. Before proving the Corollary, here is an example of its use. Prove that $\{(5, 3), (1, 4)\}$ is a basis for $\mathbb{R}^2$. Since $\dim \mathbb{R} = 2$, we just need to check that these two vectors are linearly independent (by part (c)). Since neither is a scalar multiple of the other, we are done.

Proof. (a) Here we repeat the key idea of the proof from last time showing that all bases have the same number of elements. If $S = \emptyset$, we are done. Otherwise, say $S = \{s_1, \ldots, s_k\}$ for some $k \geq 1$. We know that V has some basis $C = \{v_1, \ldots, v_n\}$. Since $V = \text{Span}(C)$, we can write

$$s_1 = a_1 v_1 + \cdots + a_n v_n.$$

Since S is linearly independent, $s_1 \neq \vec{0}$, and hence, some $a_i \neq 0$. Without loss of generality, say $a_1 \neq 0$. By the exchange lemma, we can swap s_1 for v_1 in C to get a new basis $C' = \{s_1, v_2, v_3, \ldots, v_n\}$.

If $k \geq 2$, since C' is a basis, we can write

$$s_2 = b_1 s_1 + b_2 v_2 + \cdots + b_n v_n.$$

Since s_1 and s_2 are linearly independent, at least one of $b_2, \ldots, b_n$ is nonzero. For convenience, say $b_2 \neq 0$. By the exchange lemma, the set $C'' = \{s_1, s_2, v_3, \ldots, v_n\}$ is a basis. We can repeat this process until all elements of S have been swapped into C, thus showing that $k \leq n$, as required.

(b) If $V = \text{Span}(S)$, we are done. If not, take $v \in V \setminus \text{Span}(S)$. By an earlier result, $S \cup \{v\}$ is linearly independent. We can repeat this process, but once we reach n elements, the process stops by part (a).

(c) ($\Rightarrow$) Suppose that S is linearly independent. By part (b), we can complete S to a basis B. Since $\dim V = n$, we know that $|B| = n$. So we have $S \subseteq B$ and $|S| = |B| = n$. It follows that $S = B$ is a basis, and hence, it spans V.

($\Leftarrow$) Suppose $V = \text{Span}(S)$. We saw in an earlier lecture that there is a linearly subset of S' of S with the same span as S. Since S' is linearly independent and $V = \text{Span}(S) = \text{Span}(S')$, it follows that S' is a basis, and hence $|S'| = \dim V = n$. Since $S' \subseteq S$ and $n = |S'| = |S|$, it follows that $S' = S$, and therefore, S is linearly independent.

(d) If S is infinite, there is nothing to prove. Otherwise, by removing elements from S we can find a linearly independent subset $S' \subseteq S$ with the same span. Then S' is a basis for V and hence has n elements. Since $S' \subseteq S$, we have $n = |S'| \leq |S|$.

(e) HW.

(f) HW.

Example. Prove that $\{(3,1,2), (1,0,-1), (-1,2,4), (1,3,0)\} \subset \mathbb{R}^3$ is linearly dependent.

Solution. Since $\dim \mathbb{R}^3 = 3$, a linearly independent set has at most 3 elements. $\square$

Extra time activity. Let $F = \mathbb{Z}/3\mathbb{Z}$, and consider the following twelve points in F^4:

$$\begin{array}{ccc} (1,1,2,1) & (1,1,2,0) & (2,1,2,1) \\ (1,1,0,1) & (2,0,1,0) & (1,0,1,1) \\ (2,1,1,0) & (1,2,0,0) & (1,2,2,1) \\ (1,2,0,1) & (2,0,1,1) & (0,0,2,2) \end{array}$$

Goal: find subsets of size three of this array that sum to $(0,0,0,0)$.

All solutions:

- $(1,1,2,1),(1,0,1,1),(1,2,0,1)$
- $(1,1,0,1),(1,0,1,1),(1,2,2,1)$
- $(2,1,1,0),(1,2,0,1),(0,0,2,2)$

Observations:

- Three vectors sum to zero if and only if in each component, the entries are either all the same or all different. For example, in the solution $(2,0,0,1),(2,0,1,2),(2,0,2,0)$, the entries in the first component are all 2, the entries in the second component are all 0, the entries in the third and fourth components are $0,1,2$—all different.
- If u,v,w is a solution so that $u+v+w=0$, consider the value of

  $$u+t(v-u)$$

  as t varies among the element of F. When $t=0$, we get u. When $t=1$, we get v, and when $t=2$, we get

  $$u+2(v-u) = -u+2v = -u-v = w,$$

  recalling that $2=-1$ in $F=\mathbb{Z}/3\mathbb{Z}$. We may think of $t(v-u)$ as determining a line through the origin as t varies. So then $u+t(v-u)$ is that line translated by the vector u. So finding these triples of points whose sum is zero is the same as finding lines in F^4 containing the three points.

Relation to the game Set (number-1, shading, color, shape):