Math 201 lecture for Monday, Week 4

Dimension I

Recall the following from last time:

- A set B is a *basis* for V if it
  - is linearly independent, and
  - spans V.
- If B is a basis for V, each element of V can be expressed uniquely as a linear combination of vectors in B.
- If $B = \langle v_1, \dots, v_n \rangle$ is an ordered basis for V, then the *coordinates of* $v \in V$ *with respect to* B are $(a_1, \dots, a_n)$ where

$$v = a_1 v_1 + \cdots + a_n v_n.$$

Example. Find the coordinates of $(7, -6) \in \mathbb{R}^2$ with respect to the ordered basis $B = \langle (5,3), (1,4) \rangle$.

Solution. We need to find $a, b \in \mathbb{R}$ such that

$$(7, -6) = a(5,3) + b(1,4).$$

Therefore, we solve the system of equations

$$\begin{aligned} 5a + b &= 7 \\ 3a + 4b &= -6. \end{aligned}$$

Applying our algorithm yields $a = 2$ and $b = -3$. So the coordinates of $(7, -6)$ with respect to B are given by $(2, -3)$. We write

$$[(7, -6)]_B = (2, -3).$$

Figure 1 gives the geometry. The basis vectors are in blue, and the red vectors indicate how $(7, -6)$ is a linear combination of the basis vectors.

Remark. Let $B = \langle v_1, \dots, v_n \rangle$ be an ordered basis for a vector space V. Then taking coordinates defines a bijective (why?) function

$$\begin{aligned} \phi \colon V &\to F^n \\ v &\mapsto [v]_B. \end{aligned}$$

This function has an important property: it *preserves linear structure*. By this, we mean the following: let $u, v \in V$ and let $\lambda \in F$, then we claim that

$$\phi(u + \lambda v) = \phi(u) + \lambda \phi(v). \tag{1}$$

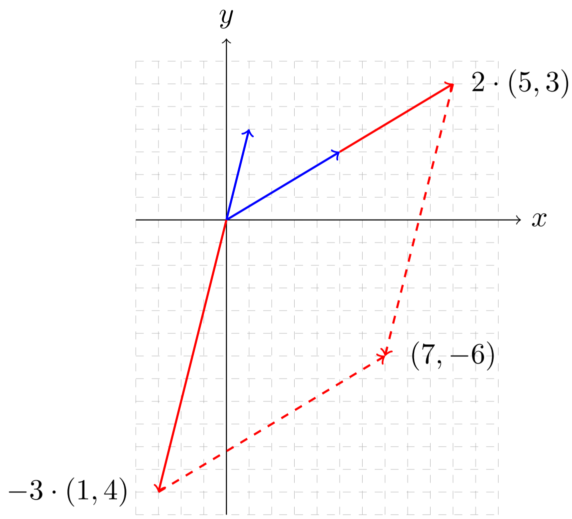

Figure 1: The coordinates of $(7, -6)$ with respect to the ordered basis $\langle(5, 3), (1, 4)\rangle$.

Note that addition and scalar multiplication happens in V on the left-hand side of this equation, and they happen in F^n on the right-hand side. The fact that ϕ is bijective and preserves linear structure means that as vector spaces V and F^n are "essentially the same". We can be more precise when we introduce linear transformations next week. For now, let us prove that equation (1) holds. We express u and v in terms of the basis:

$$\begin{aligned} u &= a_1 v_1 + \cdots + a_n v_n \\ v &= b_1 v_1 + \cdots + b_n v_n. \end{aligned}$$

It follows that

$$u + \lambda v = (a_1 + \lambda b_1)v_1 + \cdots + (a_n + \lambda b_n)v_n.$$

Then

$$\begin{aligned} \phi(u + \lambda v) &= [u + \lambda v]_B \\ &= (a_1 + \lambda b_1, \ldots, a_n + \lambda b_n) \\ &= (a_1, \ldots, a_n) + \lambda(b_1, \ldots, b_n) \\ &= [u]_B + \lambda[v]_B \\ &= \phi(u) + \lambda\phi(v). \end{aligned}$$

Definition. A vector space is *finite-dimensional* if it has a basis with a finite number of elements. If a vector space is not finite-dimensional, it is *infinite-dimensional*.

Examples. The following vector spaces are finite-dimensional:

- F^n (has a basis with n elements)
- $\mathcal{P}_d(F) = F[x]_{\leq d}$ (has a basis with $d+1$ elements)
- $M_{m\times n}$ (has a basis with $m \times n$ elements)
- $\mathbb{C}$ as a vector space over $\mathbb{R}$ (basis $\{1, i\}$).

The following are infinite-dimensional:

- $\mathcal{P}(F) = F[x]$
- $\mathbb{R}^{\mathbb{R}} = \{f \colon \mathbb{R} \to \mathbb{R}\}$
- $\{f \colon \mathbb{R} \to \mathbb{R} \colon f \text{ is continuous}\}$
- $\{f \colon \mathbb{R} \to \mathbb{R} \colon f \text{ is differentiable}\}$
- $\mathbb{R}$ as a vector space over $\mathbb{Q}$
- $\mathbb{C}$ as a vector space over $\mathbb{Q}$.

Our goal today is to show that if V is a finite-dimensional vector space, then every basis for V has the same number of elements. Thus, the following definition makes sense:

Definition. If V is a finite-dimensional vector space, then the *dimension* of V, denoted $\dim V$ or $\dim_F V$, if we want to make the scalar field explicit, is the number of elements in any of its bases.

Exchange Lemma. Suppose $B = \{v_1, \ldots, v_n\}$ is a basis for a vector space V over a field F. Further, suppose that

$$w = a_1 v_1 + \cdots + a_n v_n \in V \qquad (\star)$$

with $a_i \in F$, and such that $a_\ell \neq 0$ for some $\ell \in \{1, \ldots, n\}$. Let B' be the set of vectors obtained from B by exchanging w for v_ℓ, i.e., $B' := (B \setminus \{v_\ell\}) \cup \{w\}$. Then B' is also a basis for V.

Proof. We first show that B' is linearly independent. For ease of notation, we may assume that $\ell = 1$, i.e., that $a_1 \neq 0$. Suppose we have a linear relation among the elements of B':

$$bw + b_2 v_2 + \cdots + b_n v_n = 0$$

Substituting for w:

$$0 = b(a_1 v_1 + \cdots + a_n v_n) + b_2 v_2 + \cdots + b_n v_n = ba_1 v_1 + (ba_2 + b_2)v_2 + \cdots + (ba_3 + b_n)v_n.$$

Since the v_i are linearly independent,

$$ba_1 = ba_2 + b_2 = \cdots = ba_n + b_n = 0.$$

Since $a_1 \neq 0$, it follows that $b = 0$ and then that $b_2 = \cdots = b_n = 0$, as well. Therefore, B' is linearly independent.

We now show that B' spans V. First, solve for v_1 in $(\star)$:

$$v_1 = \frac{1}{a_1} w - \frac{a_2}{a_1} v_2 - \cdots - \frac{a_n}{a_{\ n}}.$$

To see that B' spans, take $v \in V$. Since B is a basis, v can be written as a linear combination of $B = \{v_1, \ldots, v_n\}$, but then substituting the above expression for v_1 will express v as a linear combination of $B' = \{w, v_2, \ldots, v_n\}$, as required:

$$\begin{aligned} v &= c_1 v_1 + \cdots + c_n v_n \\ &= \left(\frac{1}{a_1} w - \frac{a_2}{a_1} v_2 - \cdots - \frac{a_n}{a_{\ n}} v_n \right) + c_2 v_2 + \cdots + c_n v_n \\ &= \frac{1}{a_1} w + \left(-\frac{a_2}{a_1} + c_2 \right) v_2 + \cdots + \left(-\frac{a_n}{a_1} + c_n \right) v_n. \end{aligned}$$

□

Corollary. Suppose $B = \{v_1, \ldots, v_n\}$ is a basis for a vector space V over a field F. Further, suppose that $w \in V$ is nonzero. Then there exists $\ell \in \{1, \ldots, n\}$ such that $B' := (B \setminus \{v_\ell\}) \cup \{w\}$ is also a basis for V.

Theorem. In a finite-dimensional vector space, every basis has the same number of elements.

Proof. Let V be a finite-dimensional vector space. Among all the bases for V, let $B = \{u_1, \ldots, u_n\}$ be one of minimal size. Since B has minimal size, we know that $n = |B| \leq |C|$. Therefore C contains at least n distinct vectors $w_1, \ldots, w_n$ and possibly more. (Our goal is to show that, in fact, C contains no others.)

To take care of a trivial case, suppose $B = \emptyset$ (the case $n = 0$). In that case, we have

$$V = \mathrm{Span}(C) = \mathrm{Span}(B) = \mathrm{Span}(\emptyset) = \left\{ \vec{0} \right\}.$$

The only linearly independent set whose span is $\left\{ \vec{0} \right\}$ is $\emptyset$. So in this case, $0 = |C| = |B|$, as desired.

Now suppose that $n \geq 1$. We would again like to show that C has the same number of elements as B. The idea is to start with B, then use the exchange lemma to swap in the n elements $w_1, \ldots, w_n$ from C, one at a time, maintaining a basis at each step. To that end, let $B_0 = B$ and consider $w_1 \in C$. By the exchange lemma, we get a new basis B_1 by swapping w_1 with some $u_\ell \in B_0$. For ease of notation, let's suppose that $\ell = 1$. Therefore, $B_1 = \{w_1, u_2, \ldots, u_n\}$. Since B_1 is a basis for V, it is linearly independent and $V = \mathrm{Span}(B_1) = \mathrm{Span}(B) = \mathrm{Span}(C)$.

Next, consider $w_2 \in C$. Since B_1 is a basis, we know $w_2 \in \mathrm{Span}(B_1)$, hence, we can write

$$w_2 = a_1 w_1 + a_2 u_2 + \ldots a_n u_n$$

for some $a_i \in F$. Since w_1 and w_2 are linearly independent, at least one of $a_2, \ldots, a_n$ is nonzero. Without loss of generality, suppose $a_2 \neq 0$. Then by the exchange algorithm, $B_3 := \{w_1, w_2, u_3, \ldots, u_n\}$ is a basis. Continuing in this way, we eventually reach the basis $B_n = \{w_1, \ldots, w_n\} \subseteq C$. In fact, we must have $B_n = C$. Otherwise, there is a $w \in C \setminus B_n$. Since B_n is a basis, $w \in \mathrm{Span}(B_n)$, in other words, $w = \sum_{i=1}^{n} d_i w_i$ for some $d_i \in F$. But that can't happen since C is a basis: it's elements are linearly independent. So, in fact, C also has n elements. $\square$

Remark. If V is infinite-dimensional, it turns out that any two bases have the same cardinality. The above proof does not work to prove that, though.