Math 201 lecture for Friday, Week 4

Row and column spaces

Row rank and column rank.

Definition. Let A be an $m \times n$ matrix over F. The *row space* of A is the subspace of F^n spanned by its rows, and the *column space* of A is the subspace of F^m spanned by its columns. The *row rank* of A is the dimension of its row space, and the *column rank* of A is the dimension of its column space.

Since row operations are reversible, any matrix obtained from a matrix A by performing row operations has the same row space. In particular, the row space of A is the same as the row space of its reduced echelon form. From the structure of the reduced echelon form, it's clear that its nonzero rows form a basis for its row space. To summarize:

> **The nonzero rows of the reduced echelon form of A form a basis for the row space of A.**

This gives an algorithm for computing a basis for the row space of a matrix.

Algorithm for computing a basis for the row space and the row rank. Given an $m \times n$ matrix A, compute its reduced echelon form E. Then the rows of E are a basis for the row space of A. The number of nonzero rows in E is the row rank of A.

Example. Let

$$A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 1 & 0 \\ 7 & 8 & 2 & 4 \end{pmatrix}.$$

To compute a basis for the row space of A, compute its reduced echelon form:

$$A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 1 & 0 \\ 7 & 8 & 2 & 4 \end{pmatrix} \longrightarrow E = \begin{pmatrix} 1 & 0 & \frac{2}{3} & -4 \\ 0 & 1 & -\frac{1}{3} & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So a basis for the row space of A is:

$$\left\{ \left(1, 0, \tfrac{2}{3}, -4\right), \left(0, 1, -\tfrac{1}{3}, 4\right) \right\}.$$

Proposition. Let A be an $m \times n$ matrix with columns $A_1, \dots, A_n \in F^m$. Let $\tilde{A}$ be any matrix formed from A by performing row operations, and let $\tilde{A}_1, \dots, \tilde{A}_n \in F^m$ be its columns. Let $x_1, \dots, x_n \in F$ be any scalars. Then

$$x_1 A_1 + \cdots + x_n A_n = 0 \quad \text{if and only if} \quad x_1 \tilde{A}_1 + \cdots + x_n \tilde{A}_n = 0.$$

Proof. Write out the relation $x_1 A_1 + \cdots + x_n A_n = 0$ longhand:

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = 0.$$

Adding up the left-hand side, we see the relation is equivalent to a solution $(x_1, \dots, x_n)$ to the linear system

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\ \vdots \quad \vdots \quad &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

Or result follows since row operations do not change the set of solutions to a system of equations. $\square$

Corollary. Let E be the reduced row echelon form of a matrix A, and suppose the basic (pivot) columns have indices $j_1, \dots, j_r$. Then the columns of A indexed by $j_1, \dots, j_r$ form a basis for the column space of A.

Proof. For ease of notation, assume $j_1 = 1, j_2 = 2, \dots, j_r = r$, i.e., the first r columns of E are the pivot columns. For instance, in the case $m = 5$, $n = 7$, and $r = 3$, the matrix E would have the form

$$\begin{pmatrix} 1 & 0 & 0 & * & * & * & * \\ 0 & 1 & 0 & * & * & * & * \\ 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where the $*$s are arbitrary scalars.

Let $E_1, \dots, E_n$ denote the columns of E, and let $A_1, \dots, A_n$ denote the columns of A. It is clear that $E_1, \dots, E_r$ form a basis for the columns space of E. We need to show that $A_1, \dots, A_r$ form a basis for the columns space of A. So we need to show $A_1, \dots, A_r$ are linearly independent and span the column space of A. For linear independence, suppose that

$$x_1 A_1 + \cdots + x_r A_r = 0.$$

for some $x_i \in F$. Then, by the Proposition,

$$x_1 E_1 + \cdots + x_r E_r = 0.$$

Since $E_1, \ldots, E_r$ are linearly independent, it follows that $x_1 = \cdots = x_r = 0$, as desired. Next, to show $A_1, \ldots, A_r$ span the column space of A, it suffices to show that every other column of A is in the span. So consider a column A_j with $j > r$. Since $E_1, \ldots, E_r$ form a basis for the column space of E, we can find scalars $c_1, \ldots, c_r$ such that

$$E_j = c_1 E_1 + \cdots + c_r E_r.$$

Rewriting this equation, we get

$$c_1 E_1 + \cdots + c_r E_r - E_j = 0.$$

It then follows from the Proposition that

$$c_1 A_1 + \cdots + c_r A_r - A_j = 0,$$

which implies

$$A_j = c_1 A_1 + \cdots + c_r A_r - A_j.$$

So A_j is in the span of $A_1, \ldots, A_r$. □

We turn the Corollary into an algorithm:

Algorithm for computing a basis for the column space and the column rank. Given a matrix A, compute its reduced echelon from E. Say that columns $j_1, \ldots, j_r$ are the basic columns of E (those corresponding to the non-free variables—the one that have a single non-zero entry and that entry is equal to 1. Then columns $j_1, \ldots, j_r$ are a basis for the columns space of A. The column rank of A is r, the number of basic columns of its reduced echelon form.

WARNING: Be sure to take columns $j_1, \ldots, j_k$ of the orginal matrix, A, not of the echelon form, E. (So computing a basis for the row space is little easier, since it does not require this last step.)

Example: In the previous example, we computed the reduced echelon form of a matrix:

$$A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 1 & 0 \\ 7 & 8 & 2 & 4 \end{pmatrix} \longrightarrow E = \begin{pmatrix} 1 & 0 & \frac{2}{3} & -4 \\ 0 & 1 & -\frac{1}{3} & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The first two columns of E are its basic columns. Therefore, the first two columns of A form a basis for its column space:

$$\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 8 \end{pmatrix}.$$

NOTE: The first two columns of E in this case are the first two standard basis vectors, which clearly don't have the same span as the above two vectors.

A consequence of our discussion above is the following, rather surprising, result:

Theorem. The row rank of a matrix A is equal to its column rank.

Proof. Let E be the reduced echelon form of A. Then the number of its nonzero rows is equal to the number of its basic columns. $\square$

Definition. The *rank* of a matrix A, denoted $\mathrm{rank}(A)$ is the dimension of its row space or column space.

Suppose we have a homogeneous system of linear equations

$$\begin{aligned} a_{11}x_{11} + \cdots + a_{1n}x_n &= 0 \\ \vdots \quad\quad \vdots \quad\quad &\vdots \\ a_{m1}x_m + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

Let $A = (a_{ij})$ be the matrix of coefficients. To solve the system, we compute the reduced echelon form of the matrix A. The number of free parameters for the solution space is then the number of non-basic columns, i.e., $n - \mathrm{rank}(A)$. There is a unique solution $\vec{0}$ exactly when the reduced echelon form is the matrix with 1s along its diagonal and 0s, otherwise, i.e., exactly when there are no non-basic columns. Hence, there is only the trivial solution if and only if $\mathrm{rank}(A) = n$.

For a non-homogeneous system

$$\begin{aligned} a_{11}x_{11} + \cdots + a_{1n}x_n &= b_1 \qquad (1) \\ \vdots \quad\quad \vdots \quad\quad &\vdots \\ a_{m1}x_m + \cdots + a_{mn}x_n &= b_n. \end{aligned}$$

we would compute the echelon of the augmented matrix $[A|b]$ where b is the column with entries $b_1, \ldots, b_n$. If the system is consistent, we have seen that the set of solutions consists of any particular solution plus any vector in the span of $n - \mathrm{rank}(A)$ vectors that are solutions to the corresponding homogeneous system. So if the system is consistent, there is a unique solution if and only if $\mathrm{rank}(A) = n$.

Summary. The system (1), above, has a unique solution if and only if it is consistent and $\mathrm{rank}(A) = n$. In the case $b_1 = \cdots = b_n = 0$, the system is homogeneous and, thus, consistent ($x_1 = \cdots = x_n = 0$ is a solution). So in the homogeneous case, there is a unique solution if and only if $\mathrm{rank}(A) = n$.