Math 201 lecture for Wednesday, Week 3

Linear independence

Definition. A set $S \subset V$ is *linearly dependent* if there exist distinct[1] $u_1, \ldots, u_n \in S$, for some $n \geq 1$, and scalars $a_1, \ldots, a_n$, not all zero, such that

$$a_1u_1 + \cdots + a_nu_n = 0.$$

We call the above expression a *non-trivial dependence relation* among the u_i.

Example. The empty set is not linearly dependent.

Example. If $0 \in S$, then S is linearly dependent. For instance, $1 \cdot 0 = 0$ is a non-trivial dependence relation.

Example. Let $S = \{(1,-1,0), (-1,0,2), (-5,3,4)\} \subset \mathbb{R}^3$. Is S linearly dependent? We look for $a_1, a_2, a_3 \in \mathbb{R}$ such that

$$a_1(1,-1,0) + a_2(-1,0,2) + a_3(-5,3,4) = (0,0,0),$$

i.e., such that

$$(a_1 - a_2 - 5a_3, -a_1 + 3a_3, 2a_2 + 4a_3) = (0,0,0).$$

So we are looking for a solution to the system of linear equations

$$\begin{aligned} a_1 - a_2 - 5a_3 &= 0 \\ -a_1 + 3a_3 &= 0 \\ 2a_2 + 4a_3 &= 0. \end{aligned}$$

Apply our algorithm:

$$\left(\begin{array}{rrr|r} 1 & -1 & -5 & 0 \\ -1 & 0 & 3 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right) \xrightarrow{r_2 \to r_2 + r_1} \left(\begin{array}{rrr|r} 1 & -1 & -5 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right) \xrightarrow{r_2 \to -r_2}$$

$$\left(\begin{array}{rrr|r} 1 & -1 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right) \xrightarrow[r_3 \to r_3 - 2r_2]{r_1 \to r_1 + r_2} \left(\begin{array}{rrr|r} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Converting back to a system of equations and solving for the pivot variables gives

$$a_1 = 3a_3, \quad a_2 = -2a_3,$$

[1]Note the easily forgotten but necessary word "distinct", here.

and a_3 is arbitrary. Take $a_3 = 1$ to get the solution $a_1 = 3$, $a_2 = -2$, and $a_3 = 1$:

$$3(1, -1, 0) - 2(-1, 0, 2) + (-5, 3, 4) = (0, 0, 0).$$

Therefore, these vectors are linearly dependent.

Proposition 1. Let $S \subseteq V$. Then S is linearly dependent if and only if there exists $v \in S$ such that v is a linear combination of vectors in $S \setminus \{v\}$, i.e., if and only if $v \in \mathrm{Span}(S \setminus \{v\})$.

Proof. First note that we may assume $S \neq \emptyset$ since the empty set is not linearly dependent.

($\Rightarrow$) Suppose $a_1u_1 + \cdots + a_nu_n = 0$ for distinct $u_i \in S$ and $a_i \in F$, not all zero. Without loss of generality, we may assume that $a_1 \neq 0$. In that case, we have

$$u_1 = -\frac{a_2}{a_1}u_2 - \frac{a_3}{a_1}u_3 - \cdots - \frac{a_n}{a_1}u_n,$$

expressing u_1 as a linear combination of elements in $S \setminus \{u_1\}$. Note the special case where $S = \{0\}$. The result still holds in that case since $\{0\} = \mathrm{Span}(\emptyset)$. By definition, the empty linear combination is 0.

($\Leftarrow$) Say $v = a_1u_1 + \cdots + a_nu_n$ with distinct $u_i \in S \setminus \{v\}$ and $v \in S$. Then

$$a_1u_1 + \cdots + a_nu_n - v = 0$$

shows that S is linearly dependent. □

Definition. A set $S \subset V$ is *linearly independent* if it is not linearly dependent. This means that for all $n \geq 1$ and distinct $u_1, \ldots, u_n \in S$, if $a_1u_1 + \cdots + a_nu_n = 0$ for some $a_i \in F$, then $a_1 = \cdots = a_n = 0$. (In particular, the empty set is linearly independent.)

Remark. We say there is a *linear relation* among vectors $u_1, \ldots, u_n$ if there exist $a_i \in F$ such that $a_1u_1 + \cdots + a_nu_n = 0$. The linear relation is *trivial* if all $a_i = 0$. Thus, a subset S of V is linearly independent if every linear relation distinct elements of S is trivial.

IMPORTANT. To prove that a set of (distinct) vectors $S = \{v_1, \ldots, v_k\}$ is linearly independent start by writing the following:

> Suppose that
>
> $$a_1v_1 + \cdots + a_kv_k = 0$$
>
> for some $a_1, \ldots, a_k \in F$.

The goal is then to use some knowledge you are given about the vectors $v_1, \ldots, v_k$ to show that the relation is trivial, i.e., $a_i = 0$ for all i.

AVOID. Another way to prove that a set of vectors $S = \{v_1, \ldots, v_k\}$ is linearly independent is to suppose that some v_i is a linear combination of the vectors in $S \setminus \{v_i\}$ or to suppose that there is some nontrivial linear combination of elements in S, and then show a contradiction arises. Whenever tempted to give such a proof, check to see if the standard proof, described just above, would be clearer (as it almost always will).

Examples.

- The set $\{u\}$ is linearly independent for any nonzero $u \in V$: if $\lambda u = 0$ for some $\lambda \neq 0$, then scaling by $1/\lambda$ would give $u = 0$. But we are supposing $u \neq 0$. (Here is a case where the indirect proof of independence seems warranted.)

- $S = \{(1, -1, 0), (-1, 0, 2), (0, 1, 1)\} \subset \mathbb{R}$ is linearly independent. To see this, we follow the standard proof. Suppose that

$$a(1, -1, 0) + b(-1, 0, 2) + c(0, 1, 1) = 0,$$

which means

$$\begin{aligned} a - \ \ b \qquad &= 0 \\ -a \qquad + c &= 0 \\ 2b + c &= 0. \end{aligned}$$

Apply our algorithm (I'll just show the result of row reduction):

$$\left(\begin{array}{rrr|r} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

Thus, the only solution is $a = b = c = 0$.

- The set $S = \{1 + x, 1 + x + x^2\} \subset P_2(\mathbb{R}) = \mathbb{R}[x]_{\leq 2}$ is linearly independent. To see this, suppose that

$$a(1 + x) + b(1 + x + x^2) = 0$$

for some $a, b \in \mathbb{R}$. It follows that

$$(a + b) + (a + b)x + bx^2 = 0,$$

and, therefore, $a + b = 0$ (the coefficient the constant term or of the x-term) and $b = 0$ (the coefficient of the x^2-term). It then follows that $a = b = 0$.

Problem (leading to an important algorithm). Let

$$S = ((2,0,0),(0,1,0),(2,2,0),(0,3,1),(3,0,1))\,.$$

Find a linearly independent subset of S and write the remaining vectors as linear combinations of vectors in that subset.

Solution. Look for linear relations

$$c_1(2,0,0) + c_2(0,1,0) + c_3(2,2,0) + c_4(0,3,1) + c_5(3,0,1) = (0,0,0).$$

Convert the above relation to as system of three homogeneous linear equations in c_1, c_2, c_3, c_4, c_5 and solve:

$$\left(\begin{array}{ccccc|c} 2 & 0 & 2 & 0 & 3 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & \frac{3}{2} & 0 \\ 0 & 1 & 2 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right).$$

(Note that the first matrix has the vectors in S as columns.) So the solution space is

$$\left(-c_3 - \frac{3}{2}c_5, -2c_3 + 3_5, c_3, -c_5, c_5 : c_3, c_5 \in \mathbb{R}\right),$$

or, in parametric form

$$\left\{ c_3 \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_5 \begin{pmatrix} -\frac{3}{2} \\ 3 \\ 0 \\ -1 \\ 1 \end{pmatrix} : c_3, c_5 \in \mathbb{R} \right\}.$$

Let T be the set of columns in our original matrix with the same indices as those for the non-free (i.e., pivot or leading) variables in the row-reduced matrix. In other words,

$$T = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right\}.$$

We claim that T is linearly independent. Suppose there is a linear relation (switching to row notation for convenience):

$$a(2,0,0) + b(0,1,0) + c(0,3,1) = 0.$$

To show that $a = b = c = 0$ is the only solution, we convert to a matrix and row-reduce as usual:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore, we must have $a = b = c = 0$, as claimed. **Important:** In fact, there was no need to do that last computation since we have already done it. To see that, go back to our original row-reduction

$$\left(\begin{array}{ccccc|c} 2 & 0 & 2 & 0 & 3 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & \frac{3}{2} & 0 \\ 0 & 1 & 2 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

and only pay attention to the first, second, and fourth columns. So the verification that T is linearly independent was secretly guaranteed by its construction.

It remains to be shown that the remaining columns (those corresponding to the free variables), i.e., $(2, 2, 0)$ and $(3, 0, 1)$, in row notation) are in the span of T. We have found all solutions to

$$c_1(2, 0, 0) + c_2(0, 1, 0) + c_3(2, 2, 0) + c_4(0, 3, 1) + c_5(3, 0, 1) = (0, 0, 0) \qquad (1)$$

and found that c_3 and c_5 are free variables. To see that $(2, 2, 0)$ is in the span of T, find the solution to our system for which $(c_3, c_5) = (1, 0)$, then solve for $(2, 2, 0)$ in (1). The solution in this case is

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Therefore,

$$-(2, 0, 0) - 2(0, 1, 0) + 1 \cdot (2, 2, 0) + 0 \cdot (0, 3, 1) + 0 \cdot (3, 0, 1) = (0, 0, 0),$$

and, thus,

$$(2, 2, 0) = (2, 0, 0) + 2(0, 1, 0).$$

Similarly, to show $(3, 0, 1)$ is in the span of T, we set $(c_3, c_5) = (0, 1)$. The corresponding solution is

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} \\ 3 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

Therefore,

$$-\frac{3}{2}(2,0,0)+3(0,1,0)+0\cdot(2,2,0)-1\cdot(0,3,1)+1\cdot(3,0,1)=(0,0,0),$$

Solving for $(3,0,1)$ gives

$$(3,0,1)=\frac{3}{2}(2,0,0)-3(0,1,0)+(0,3,1).$$

We summarize the underlying important algorithm: Let $S=\{v_1,\ldots,v_k\}\in F^n$. To find a linearly independent subset T of S such that $\mathrm{Span}(T)=\mathrm{Span}(S)$:

- Let M be the matrix with *columns* $v_1,\ldots,v_k$.
- Compute M', the row-reduced form of M.
- Let $j_1,\ldots,j_d$ be the indices of the pivot columns of M' (the ones containing the leading 1s).
- Set $T=\{v_{j_1},\ldots,v_{j_d}\}$.

Note: The set T is a subset of the columns of M *not* of M'!

The elements of $S\backslash T$ correspond to the free variables, and we can write these elements as linear combinations of the elements of T by setting each free variable in turn equal to 1 and setting the remaining free variables equal to 0.

We end with a result of fundamental importance:

Theorem. Let $S\subseteq V$ be linearly independent, and let $v\in\mathrm{Span}(S)$. Then v has a unique expression as a linear combination of elements of S. In other words, if $v=\sum_{i=1}^{k}a_iu_i$ and $v=\sum_{i=1}^{\ell}b_iw_i$ for some nonzero $a_i,b_i\in F$ and some distinct $u_i\in S$ and distinct $w_i\in S$, then up to re-indexing, we have $k=\ell$, $u_i=w_i$, and $a_i=b_i$ for all i.

Proof. Say $v=\sum_{i=1}^{n}a_iu_i$ and $v=\sum_{i=1}^{n}b_iu_i$ for some $a_i,b_i\in F$ and $u_i\in S$. (By letting some a_i and b_i equal zero, these expressions represent two arbitrary representations of v as linear combinations of elements of S, i.e., we can use the same u_i and n for both expressions.) It follows that

$$0=v-v=\sum_{i=1}^{n}a_iu_i-\sum_{i=1}^{n}b_iu_i=\sum_{i=1}^{n}(a_i-b_i)u_i.$$

Since S is linearly independent, it follows that $a_i-b_i=0$ for all i. The result follows. $\square$

Example. The previous result does not hold if S is linearly dependent. For instance, consider the set $S = \{(1,1),(2,2)\} \subset \mathbb{R}$. Then

$$(3,3) = (1,1) + (2,2) = 2(1,1) + \frac{1}{2}(2,2) = 3(1,1) + 0(2,2) = \text{ etc.}$$

Exercise. Prove that the converse of the previous proposition holds: if each element of $\operatorname{Span} S$ can be expressed uniquely as a linear combination of elements of S, then S is linearly independent.