Math 201 lecture for Monday, Week 3

Subspaces and spanning sets II

In today's lecture, we start by proving a simple (but useful) results about spanning sets. We then present several examples of subspaces and spanning sets.

Recall the definitions presented last time:

Definition. Let S be a nonempty subset of V. Then $v \in V$ is a *linear combination* of vectors in S if there exist $u_1, \ldots, u_n \in S$ and $a_1, \ldots, a_n \in F$ (for some n) such that

$$v = \sum_{i=1}^{n} a_i u_i = a_1 u_1 + \cdots + a_n u_n.$$

Definition. Let S be a nonempty subset of V. The *span* of S, denoted $\operatorname{Span}(S)$, is the set of all linear combinations of elements of S. By convention $\operatorname{Span} \emptyset := \{0\}$, and we say that 0 is the *empty linear combination*.

Lemma. Let V be a vector space over F, let $S \subseteq V$, and let $v \in V$. Then

$$\operatorname{Span}(S \cup \{v\}) = \operatorname{Span}(S) \quad \Leftrightarrow \quad v \in \operatorname{Span}(S).$$

Proof. ($\Rightarrow$) If $\operatorname{Span}(S \cup \{v\}) = \operatorname{Span}(S)$, then since $v \in \operatorname{Span}(S \cup \{v\})$, it follows that $v \in \operatorname{Span}(S)$.

($\Leftarrow$) Suppose that $v \in \operatorname{Span}(S)$. We wish to show that $\operatorname{Span}(S \cup \{v\}) = \operatorname{Span}(S)$. Suppose that $w \in \operatorname{Span}(S \cup \{v\})$. Then we can write

$$w = a_1 s_1 + \cdots + a_k s_k + bv$$

for some $s_1, \ldots, s_k \in S$ and some $a_1, \ldots, a_k, b \in F$. We are given that $v \in \operatorname{Span}(S)$. Hence,

$$v = c_1 t_1 + \cdots + c_\ell t_\ell$$

for some $t_1, \ldots, t_\ell \in S$ and some $c_1, \ldots, c_\ell \in F$. Substituting into the previous equation, we see

$$\begin{aligned} w &= a_1 s_1 + \cdots + a_k s_k + b(c_1 t_1 + \cdots + c_\ell t_\ell) \\ &= a_1 s_1 + \cdots + a_k s_k + bc_1 t_1 + \cdots + bc_\ell t_\ell \\ &\in \operatorname{Span}(S). \end{aligned}$$

We have shown that $\operatorname{Span}(S \cup \{v\}) \subseteq \operatorname{Span}(S)$. The opposite inclusion also holds since one is easily sees that

$$S \subseteq S \cup \{v\} \Rightarrow \operatorname{Span}(S) \subseteq \operatorname{Span}(S \cup \{v\}).$$

$\square$

We now move on to examples of subspaces and spanning sets.

Example. Recall from the reading that $P_k(F)$ is the vector space of polynomials of degree at most k with coefficients in F. Another, more standard, notation for this vector space is $F[x]_{\leq k}$. We have that

$$P_k(F) = F[x]_{\leq 2} = \mathrm{Span}\{1, x, \dots, x^k\}.$$

Now let

$$S = \left\{x^2 + 3x - 2, 2x^2 + 5x - 3\right\} \subset \mathbb{R}[x]_{\leq 2}.$$

Is $-x^2 - 4x + 4 \in \mathrm{Span}(S)$?

Solution. We are looking for $a, b \in \mathbb{R}$ such that

$$-x^2 - 4x + 4 = a(x^2 + 3x - 2) + b(2x^2 + 5x - 3),$$

in other words, such that

$$-x^2 - 4x + 4 = (a + 2b)x^2 + (3a + 5b)x + (-2a - 3b).$$

So we need to see if the following system of linear equations has a solution:

$$\begin{aligned} a + 2b &= -1 \\ 3a + 5b &= -4 \\ -2a - 3b &= 4. \end{aligned}$$

Applying Gaussian elimination, we find

$$\left(\begin{array}{rr|r} 1 & 2 & -1 \\ 3 & 5 & -4 \\ -2 & -3 & 4 \end{array}\right) \quad \rightsquigarrow \quad \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right).$$

We see that the system is inconsistent, i.e., it has no solutions. So $-x^2 - 4x + 4 \notin \mathrm{Span}(S)$.

Definition. Let S be any set, and consider the function space $F^S := \{f : S \to F\}$. For each $s \in S$, define the *characteristic function* $\chi_s \in F^S$ for s by

$$\begin{aligned} \chi_s \colon S &\to F \\ t &\mapsto \begin{cases} 1 & \text{if } t = s \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Example. Let $S = \{1, 2, 3\}$, and consider the function $f : S \to \mathbb{R}$ given by $f(1) = -1$, $f(2) = \pi$, and $f(3) = 16$. Then we can write f as a linear combination of characteristic functions:

$$f = -\chi_1 + \pi\chi_2 + 16\chi_3.$$

For instance,

$$\begin{aligned} f(2) &= (-\chi_1 + \pi\chi_2 + 16\chi_3)(2) \\ &= -\chi_1(2) + \pi\chi_2(2) + 16\chi_3(2) \\ &= -0 + \pi \cdot 1 + 16 \cdot 0 = \pi. \end{aligned}$$

In this way, if S is finite, then $\{\chi_s : s \in S\}$ generates F^S. On the other hand, if S is infinite, things are more complicated. For instance, consider the case where $S = \mathbb{N} = \{0, 1, 2, \dots\}$. Then $\mathbb{R}^{\mathbb{N}}$ is the vector space of infinite real sequences. For instance, the sequence $1, 1/2, 1/4, /8, \dots$ is the function $f \in \mathbb{R}^{\mathbb{N}}$ given by $f(i) := 1/2^i$. If we try to write f as a linear combination of characteristic functions, we would have

$$f = \chi_0 + \frac{1}{2}\chi_1 + \frac{1}{4}\chi_2 + \frac{1}{8}\chi_3 + \cdots,$$

an infinite sum. However, by definition, the span of a set is the collection of all *finite* linear combinations of elements in the set. Infinite linear combinations like those above involve questions of convergence, and we are not concerned with those issues at the moment.

Definition. A linear equation of the form $a_1x_1 + \cdots + a_nx_n = 0$ where $a_i \in F$ is called *homogeneous.*

Proposition. The solution set to a system of homogeneous linear equations in n unknowns and with coefficients in F is a subspace of F^n.

Proof. First note that the zero vector satisfies any homogeneous linear equation. So the solution set is nonempty. Next, let $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ be solutions to a system of homogeneous linear equations, and let $a_1x_1 + \cdots + a_nx_n = 0$ be any equation in the system. Thus,

$$\begin{aligned} a_1u_1 + \cdots + a_nu_n &= 0 \\ a_1v_1 + \cdots + a_nv_n &= 0. \end{aligned}$$

Now let $\lambda \in F$ and consider

$$u + \lambda v = (u_1 + \lambda v_1, \dots, u_n + \lambda v_n).$$

The following calculation shows that $u + \lambda v$ is also a solution to the equation

$$\begin{aligned} a_1(u_1 + \lambda v_1) + \cdots + a_n(u_n + \lambda v_n) &= a_1 u_1 + \cdots + a_n u_n + \lambda(a_1 v_1 + \cdots + a_n v_n) \\ &= 0 + \lambda \cdot 0 = 0. \end{aligned}$$

□

Terminology. Since the solution set to a system of homogeneous linear equations is a subspace, we usually refer to the solution set as the *solution space* for the system.

Example. Writing a solution to a system of homogeneous linear equations in vector form yields a set of generators for the solution space. For example, consider the system

$$\begin{aligned} x \quad\quad + z + w &= 0 \\ 2x + y \quad\quad - w &= 0 \\ 3x + y + z \quad\quad &= 0. \end{aligned}$$

We solve the system by performing Gaussian elimination (intermediary steps omitted):

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & -1 & 0 \\ 3 & 1 & 1 & 0 & 0 \end{array}\right) \quad \rightsquigarrow \quad \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Converting back into equations and solving for the leading (pivot) variables gives

$$\begin{aligned} x &= -z - w \\ y &= 2z + 3w. \end{aligned}$$

So the set of solutions (in parametric form) is

$$\{(-z - w, 2z + 3w, z, w) : z, w \in \mathbb{R}\},$$

or, written in vector form,

$$\left\{ z \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} -1 \\ 3 \\ 0 \\ 1 \end{pmatrix} : z, w \in \mathbb{R} \right\} = \operatorname{Span} \left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

The solution space is generated by two vectors.