Math 201 lecture for Friday, Week 3

Bases

Definition. A subset $B \subset V$ is a *basis* if it is linearly independent and spans V. An *ordered basis* is a basis whose elements have been listed as a sequence: $B = \langle b_1, b_2, \dots \rangle$.[1]

Warning: Our book defines a *basis* to be what we are calling an *ordered basis*. That's not standard, and there are problems with that idea when talking about infinite-dimensional vector spaces, which we will not go into here. We will, however, use the book's notation of "$\langle$" and "$\rangle$" to denote an ordered basis. Thus, for us, the word *basis* will mean "unordered basis", and we will try to be careful to say "ordered basis" when relevant (but will sometimes forget).

Examples.

(a) The *standard ordered basis* for F^n is $\langle e_1, \dots, e_n \rangle$ where the i-th standard basis vector is $e_i = (0, \dots, 0, 1, 0, \dots, 0)$, the vector with i-th component 1 and all other components 0. For instance, the standard ordered basis for F^3 is

$$\langle (1,0,0), (0,1,0), (0,0,1) \rangle.$$

Here is another possible ordered basis for F^3:

$$\langle (1,0,0), (0,1,0), (1,1,1) \rangle.$$

Exercise: check that the above vectors are linearly independent and span F^3.

(b) One ordered basis for the vector space $P_3(F) = F[x]_{\leq 3}$ of polynomials of degree most three is

$$\langle 1, x, x^2, x^3 \rangle.$$

(c) One ordered basis for, $M_{2\times 3}(F)$, the vector space of 2×3 matrices over a field F, is

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ M_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ M_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ M_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ M_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

[1] Every vector space has a basis—we will prove this in the finite-dimensional case. An infinite-dimensional vector space may not have a countable basis, i.e., one that can be indexed by the natural numbers. There is a link to a supplemental article at our course homepage, if you would like to know more.

These matrices span $M_{2\times 3}(F)$:

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = aM_1 + bM_2 + cM_3 + dM_4 + eM_5 + fM_6.$$

To see they are linearly independent, suppose the above sum is 0, i.e., the zero matrix. Then we must have $a = b = c = d = e = f = 0$.

Last time, we showed the following proposition:

Proposition 1 from previous lecture. Let $S \subseteq V$. Then S is linearly dependent if and only if there exists $v \in S$ such that v is a linear combination of vectors in $S \setminus \{v\}$, i.e., if and only if $v \in \text{Span}(S \setminus \{v\})$.

We use this result to prove the following:

Proposition 1. Any finite subset S of V has a linearly independent subset with the same span. In other words, if S is a finite set, then there is a subset of S that is a basis for $\text{Span}(S)$.

Proof. If S is linearly independent, we are done. If not, then by Proposition 1 from the previous lecture, there exists $v \in S$ such that $v \in \text{Span}(S \setminus \{v\})$. It follows that $\text{Span}(S) = \text{Span}(S \setminus \{v\})$. If $S \setminus \{v\}$ is linearly independent, we are done. If not, repeat the above step. The process will end eventually since S is finite. We are OK even if the process ends at the empty set since the empty set is linearly independent. (For instance, if $S = \{0\}$, our process would end at $\emptyset$.) $\square$

In the above, we create a basis for $\text{Span}(S)$ by discarding elements of S. Another possibility is to start at the empty set and start adding elements S that are linearly independent of those we have so far. This follows from:

Proposition 2. If $T \subset V$ is linearly independent and $v \in V \setminus T$, then $T \cup \{v\}$ is linearly dependent if and only if $v \in \text{Span}(T)$.

Proof. ($\Rightarrow$) Suppose that $v \in V \setminus \{v\}$ and that $T \cup \{v\}$ is linearly dependent. Then we may write

$$av + a_1u_1 + \cdots + a_nu_n = 0 \qquad (\star)$$

for some $a, a_1, \ldots, a_n \in F$, *not all zero*, and distinct $u_i \in T$. We can always assume that v appears in this expression by taking $a = 0$, if necessary. But, in fact, $a \neq 0$ since otherwise $(\star)$ would be a linear relation among distinct elements of T. Since T is linearly independent, this would mean that all the $a_i = 0$, in addition to $a = 0$. However, we know that at least one of these scalars in nonzero.

Thus, it must be that $a \neq 0$. We can then solve for v in ($\star$):

$$v = -\frac{a_i}{a}u_1 - \cdots - \frac{a_n}{a}u_n \in \mathrm{Span}(T).$$

($\Leftarrow$) Suppose that $v \in \mathrm{Span}(T)$. Then

$$v = a_1u_1 + \cdots + a_nu_n$$

for some $a_i \in F$ and $u_i \in T$. Since $v \notin T$, it follows that

$$a_1u_1 + \cdots + a_nu_n + (-1) \cdot v = 0$$

is a nontrivial relation among elements of $T \cup \{v\}$. So $T \cup \{v\}$ is linearly dependent. $\square$

Alternate proof of Proposition 1. We are starting with a finite set S and looking for a subset T of S that is linearly independent and generates $V = \mathrm{Span}(S)$. If $S = \emptyset$ or $S = \{0\}$, we take $T = \emptyset$ and are done. If not, there exists a nonzero element $u_1 \in S$, and we set $T = \{u_1\}$. If $\mathrm{Span}(T) = \mathrm{Span}(S)$, we are done. If not, then there exists $u_2 \in S$ such that $u_2 \notin \mathrm{Span}(T)$. We then append u_2 to T. So now $T = \{u_1, u_2\}$, and by Proposition 2, the set T is linearly independent. If $\mathrm{Span}(T) \neq \mathrm{Span}(S)$, repeat to find $u_3 \in S$ linearly independent of u_1 and u_2. Etc. Since S is finite, the process eventually stops. $\square$

Example. Let $V = (\mathbb{Z}/3\mathbb{Z})^3$, a vector space over $\mathbb{Z}/3\mathbb{Z}$.

How many elements are in V? A point in V has the form (x_1, x_2, x_3), and there are 3 choices for each x_i. Hence, the number of elements in V is $|V| = 3^3 = 27$.

As an exercise, check that the following is a subspace of V:

$$W = \{(x_1, x_2, x_3) \in V : x_1 + x_2 + x_3 = 0\}.$$

How many elements are in W? We have,

$$W = \{(-x_2 - x_3, x_2, x_3) : x_2, x_3 \in \mathbb{Z}/3\mathbb{Z}\}.$$

As we let x_2 and x_3 vary, we get 9 elements:

$$\{(0,0,0), (2,1,0), (1,2,0), (2,0,1), (1,1,1), (0,2,1), (1,0,2), (0,1,2), (2,2,2)\}.$$

Let's try to find a linearly independent generating set for W. Start with $v_1 := (2,1,0)$. The span of $\{v_1\}$ has three elements:

$$\begin{aligned} 0 \cdot (2,1,0) &= (0,0,0) \\ 1 \cdot (2,1,0) &= (2,1,0) \\ 2 \cdot (2,1,0) &= (1,2,0). \end{aligned}$$

Next, note that $v_2 = (2, 0, 1)$ is not in $\mathrm{Span}(\{v_1\})$. By Proposition 2, we see that $S := \{v_1, v_2\}$ is linearly independent. We claim $\mathrm{Span}(S) = W$. First, since $v_1, v_2 \in W$, we see $\mathrm{Span}(S) \subseteq W$. Next, by Theorem 1, every element of $\mathrm{Span}(S)$ has a unique expression of the form

$$a_1 v_1 + a_2 v_2$$

where $a_1, a_2 \in \mathbb{Z}/3\mathbb{Z}$. Hence, $|\mathrm{Span}(S)| = 3^2 = 9$. Since $\mathrm{Span}(S) \subseteq W$ and $|\mathrm{Span}(S)| = |W| = 9$, it follows that $\mathrm{Span}(S) = W$.

Proposition 3. If B is a basis for V, then every element of V can be expressed uniquely as a linear combination of elements of B.

Proof. Since B is linearly independent, we've already seen that every element in $\mathrm{Span}(B)$ can be written uniquely as a linear combination of elements of B. Since B is a basis, $\mathrm{Span}(B) = V$. $\square$

Definition. Let $B = \langle v_1, \ldots, v_n \rangle$ be an ordered basis for V. Given $v \in V$, there are unique $a_1, \ldots, a_n \in F$ such that

$$v = a_1 v_1 + \cdots + a_n v_n.$$

The *coordinates of v with respect to the basis B* are the components of the vector $(a_1, \ldots, a_n) \in F^n$. We write

$$[v]_B = (a_1, \ldots, a_n).$$

Examples.

(a) Let $v = (x, y, z) \in F^3$. The coordinates of v with respect to the standard ordered basis $B = \langle e_1, e_2, e_3 \rangle$ are (x, y, z) since

$$(x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1) = xe_1 + ye_2 + ze_3.$$

Now consider $B' = \langle (1, 0, 0), (1, 1, 0), (1, 1, 1) \rangle$. Then the coordinates of v with respect to B' are $(x - y, y - z, z)$ since

$$(x, y, z) = (x - y)(1, 0, 0) + (y - z)(1, 1, 0) + z(1, 1, 1).$$

(b) Recall the ordered basis $\langle M_1, \ldots, M_6 \rangle$ for $M_{2\times 3}(F)$ defined earlier. Then the coordinates of the matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

are $(a, b, c, d, e, f) \in F^6$.