Math 201 lecture for Wednesday, Week 2

Vector spaces

Let F be a field, e.g., $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}/2\mathbb{Z}$ (but not $\mathbb{Z}$).

Definition. A *vector space over F* is a set V with two operations

$$\text{vector addition:} \qquad \begin{aligned} +\colon V \times V &\to V \\ (v, w) &\mapsto v + w \end{aligned}$$

$$\text{scalar multiplication:} \qquad \begin{aligned} +\colon F \times V &\to V \\ (a, v) &\mapsto av \end{aligned}$$

such that the following hold for all $x, y, z \in V$ and $a, b \in F$:

1. $x + y = y + x$ (commutativity of addition).
2. $(x + y) + z = (x + y) + z$ (associativity of addition).
3. There exists $0 \in V$ such that $0 + w = w$ for all $w \in V$.
4. There exists $-x \in V$ such that $x + (-x) = 0$.
5. For $1 \in F$, we have $1 \cdot x = x$.
6. $(ab)x = a(bx)$ (associativity of scalar multiplication).
7. $a(x + y) = ax + ay$ (distributivity).
8. $(a + b)x = ax + bx$ (distributivity).

Remark. Rules 1–4 provide the *additive structure* and say that under addition V forms an *abelian group*. Rules 5–8 deal with the second operation, scalar multiplication. Together, they provide a *linear structure* for the set V.

Exercise. Let v be an element of a vector space. Prove that $(-1)v = -v$.

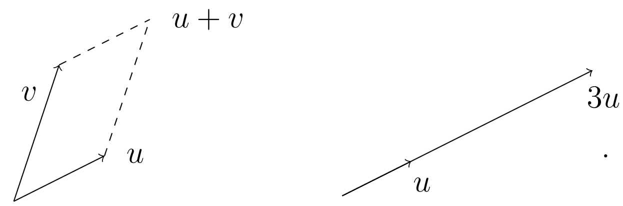
Example. Let $F^n = \underbrace{F \times \cdots \times F}_{n \text{ times}} = \{(a_1, \ldots, a_n) : a_i \in F \text{ for } i = 1, \ldots, n\}$ with the operations

$$\begin{aligned} &(a_1, \ldots, a_n) + (b_1, \ldots, b_n) := (a_1 + b_1, \ldots, a_n + b_n) \\ &c(a_1, \ldots, a_n) := (ca_1, \ldots, ca_n) \end{aligned}$$

for all $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in F^n$ and $c \in F$. Then F^n is a vector space.

Special cases:

(a) $F = \mathbb{R}$ and $n = 2$. This gives $\mathbb{R}^2$ with its usual linear structure. Addition is given by the "parallelogram rule" and scalar multiplication scales length:



Here are examples of the vector space axioms in the special case $V = \mathbb{R}^2$:

1. commutativity of $+$:
$$(6, 3) + (-2, 4) = (4, 7) = (-2, 4) + (6, 3);$$

2. associativity of $+$:
$$\begin{aligned} ((6, 3) + (-2, 4)) + (0, 2) &= (4, 7) + (0, 2) \\ &= (4, 9) \\ &= (6, 3) + (-2, 6) \\ &= (6, 3) + ((-2, 4) + (0, 2)); \end{aligned}$$

3. zero vector:
$$(0, 0) + (6, 3) = (6, 3);$$

4. additive inverses:
$$(6, 3) + (-6, -3) = (0, 0);$$

5. scaling by 1:
$$1 \cdot (6, 3) = (1 \cdot 6, 1 \cdot 3) = (6, 3);$$

6. associativity of scalar multiplication:
$$(3 \cdot 2)(6,3) = 6(6,3) = (36,18) = 3(12,6) = 3\left(2\left(6,3\right)\right);$$
7. distributivity:
$$3((6,3) + (-2,4)) = 3(4,7) = (12,21)$$
and
$$3(6,3) + 3(-2,4) = (18,9) + (-6,12) = (12,21);$$
8. distributivity:
$$(3+2)(6,3) = 5(6,3) = (30,15)$$
and
$$3(6,3) + 2(6,3) = (18,9) + (12,6) = (30,15).$$

(b) $F = \mathbb{Z}/3\mathbb{Z}$ and $n = 4$. For example, $(0,1,0,0), (1,1,0,2) \in (\mathbb{Z}/3\mathbb{Z})^4$, and
$$(0,1,0,0) + 2(1,1,0,2) = (0,1,0,0) + (2,2,0,1) = (2,0,0,1).$$

(c) The field F is a vector space over itself (this is the case of F^n with $n = 1$).

More examples of vector spaces.

(i) The field $\mathbb{C}$ is a vector space over $\mathbb{R}$. For all $a, b, c, d, t \in \mathbb{R}$, we have
$$\begin{aligned}&(a+bi) + (c+di) = (a+c) + (b+d)i\\ &t(a+bi) = ta + (tb)i.\end{aligned}$$

(ii) The field $\mathbb{R}$ is a vector space over $\mathbb{Q}$.

(iii) The set of $m \times n$ matrices with entries in F:
$$M_{m\times n} := \left\{ \begin{pmatrix} a_{11} & \dots & a_{1n} \\ & \vdots & \\ a_{m1} & \dots & a_{mn} \end{pmatrix} : a_{ij} \in F \text{ for all } i, j \right\}.$$
has a standard vector space structure. Given $A \in M_{m\times n}$, denote the entry in its i-the row and j-th column by A_{ij}. Define the vector space operations on $M_{m\times n}$ as follows:

addition: $(A+B)_{ij} := A_{ij} + B_{ij}$ for all $A, B \in M_{m\times n}$;

scalar multiplication: $(cA)_{ij} := cA_{ij}$ for all $A \in M_{m\times n}$ and $c \in F$.

For example, let $F = \mathbb{Q}$, $m = 2$, and $n = 3$. We have

$$2\begin{pmatrix} 1 & 0 & 3 \\ -1 & 2 & 0 \end{pmatrix} + 5\begin{pmatrix} 0 & 2 & -1 \\ 1 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 10 & 1 \\ 3 & 4 & 20 \end{pmatrix}.$$

Calling this last matrix A, we have $A_{1,1} = 2$, $A_{1,2} = 10, \ldots, A_{2,3} = 20$.

(iv) (**Important.**) If S is any set, let F^S be the set of functions $f\colon S \to F$. This function space is naturally an F-vector space (i.e., a vector space with scalar field F) with the following operations: for $f, g \in F^S$ and $t \in F$ define $f + g$ and tf by

addition: $$(f + g)(s) := f(s) + g(s)$$

scalar multiplication: $$(tf)(s) := t(f(s)).$$

Special cases:

- If $S = \{1, \ldots, n\}$, then F^S is essentially F^n. For example, we can think of $(3, 2) \in \mathbb{R}$ as the function

$$\begin{aligned} f\colon \{1,2\} &\to \mathbb{R} \\ 1 &\mapsto 3 \\ 2 &\mapsto 2. \end{aligned}$$

  In general, $(a_1, \ldots, a_n) \in F^n$ can be thought of as the function

$$\begin{aligned} f\colon \{1, \ldots, n\} &\to F \\ i \quad &\mapsto a_i. \end{aligned}$$

- Similarly, if $S = \{(i, j) : i = 1, \ldots, m \text{ and } j = 1, \ldots, n\}$, then F^S may be identified with $M_{m\times n}$ with $f \in F^S$ corresponding to the matrix A where $A_{ij} = f(i, j)$.
- If $S = \{1, 2, 3, \ldots\}$, then F^S is the vector space of infinite sequences in F. For example, the sequence $1, 1/2, 1/4, 1/8, \ldots$ in $\mathbb{Q}$ can be identified with the function $f\colon S \to \mathbb{Q}$ defined by $f(i) = 1/2^i$.

Definition. A subset $W \subseteq V$ of a vector space V is a *subspace* of V is it is a vector space with the operations of addition and scalar multiplication inherited from V.

We will talk about subspaces in the next class.