Math 201 lecture for Friday, Week 2

Subspaces and spanning sets I

Note: Unless otherwise stated, from now on V will denote a vector space over a field F.

The over-arching goal of the next several classes is to define the notions of *dimension* and *isomorphism* and show that every finite-dimensional vector space over F is isomorphic to the vector space of d-tuples, F^d, where d is the dimension. Today's class lays some of the groundwork for reaching that goal.

Definition. Let S be a nonempty subset of V. Then $v \in V$ is a *linear combination* of vectors in S if there exist $u_1, \ldots, u_n \in S$ and $a_1, \ldots, a_n \in F$ (for some n) such that

$$v = \sum_{i=1}^{n} a_i u_i = a_1 u_1 + \cdots + a_n u_n.$$

Example. Let $S = \{(3,2), (2,-1)\} \subset \mathbb{R}$. Is $(-1,4)$ a linear combination of vectors in S? In other words, do there exist $a, b \in \mathbb{R}$ such that

$$a(3,2) + b(2,-1) = (-1,4)?$$

Since $a(3,2) + b(2,-1) = (3a + 2b, 2a - b)$, the above requirement is equivalent to the existence of $a, b \in \mathbb{R}$ such that

$$\begin{aligned} 3a + 2b &= -1 \\ 2a - b &= 4, \end{aligned}$$

a system of linear equations! Apply our algorithm to look for solutions:

$$\left(\begin{array}{rr|r} 3 & 2 & -1 \\ 2 & -1 & 4 \end{array}\right) \xrightarrow{r_1 \to r_1 - r_2} \left(\begin{array}{rr|r} 1 & 3 & -5 \\ 2 & -1 & 4 \end{array}\right) \xrightarrow{r_2 \to r_2 - 2r_1} \left(\begin{array}{rr|r} 1 & 3 & -5 \\ 0 & -7 & 14 \end{array}\right) \xrightarrow{r_2 \to -\frac{1}{7} r_2}$$

$$\left(\begin{array}{rr|r} 1 & 3 & -5 \\ 0 & 1 & -2 \end{array}\right) \xrightarrow{r_1 \to r_1 - 3r_2} \left(\begin{array}{rr|r} 1 & 0 & 1 \\ 0 & 1 & -2 \end{array}\right).$$

Thus, $a = 1$ and $b = -2$. Check:

$$1 \cdot (3,2) - 2(2,-1) = (-1,4). \quad \checkmark$$

So $(-1,4)$ is a linear combination of the two given vectors. (If it were not, we would have had an inconsistent system, i.e., a system with no solutions.)

Definition. Let S be a nonempty subset of V. The *span* of S, denoted $\operatorname{Span}(S)$, is the set of all linear combinations of elements of S. By convention $\operatorname{Span}\emptyset := \{0\}$, and we say that 0 is the *empty linear combination.*

Example. In $\mathbb{R}^2$,

$$\operatorname{Span}\{(1,1)\} = \{(a,a) : a \in \mathbb{R}\}.$$

In $\mathbb{R}^3$,

$$\operatorname{Span}\{(1,0,0),(0,1,0)\} = \{a(1,0,0)+b(0,1,0) : a,b \in \mathbb{R}\} = \{(a,b,0) : a,b \in \mathbb{R}\}.$$

Note that the same set can be spanned by different sets of vectors, for instance,

$$\begin{aligned}\operatorname{Span}\{(1,0,0),(0,1,0)\} &= \operatorname{Span}\{(1,0,0),(0,2,0)\} \\ &= \operatorname{Span}\{(1,0,0),(0,1,0),(2,3,0)\}.\end{aligned}$$

A point in $\mathbb{R}^3$ is in any of these sets if and only if its third component is 0.

Definition. A subset $W \subseteq V$ is a *subspace* of V if W is a vector space itself with the operations of addition and scalar multiplication inherited from V.

Proposition. $W \subseteq V$ is a subspace of V if and only if

1. $0 \in W$
2. W is closed under addition ($x, y \in W \Rightarrow x + y \in W$)
3. W is closed under scalar multiplication ($c \in F$ and $w \in W \Rightarrow cw \in W$).

Proof. Exercise. Part 1 is there to ensure that W is nonempty. (Note that Part 2 and Part 3 are vacuously true for the empty set, and yet the empty set is not a subspace because of Part 1.) $\square$

Examples.

1. $W = \{(a,0) : a \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$.

   Proof. Letting $a = 0$, we see $(0,0) \in W$. If $(a,0),(b,0) \in W$, then $(a,0)+(b,0) = (a+b,0) \in W$. If $c \in \mathbb{R}$ and $(a,0) \in W$, then $c(a,0) = (ca,0) \in W$. Thus, W is a subspace of $\mathbb{R}^2$. $\square$

2. Let

$$\begin{aligned}V &= \{f\colon \mathbb{R} \to \mathbb{R} : f \text{ is continuous}\}, \\ W &= \{f\colon \mathbb{R} \to \mathbb{R} : f \text{ is differentiable}\}.\end{aligned}$$

Both V and W are subspaces of the vector space $\mathbb{R}^{\mathbb{R}}$ of functions from $\mathbb{R}$ to $\mathbb{R}$ (recall our earlier notation F^S for functions from a set S to a field F), and W is a subspace of V.

3. Let $W = \{(a, b) \in \mathbb{R}^2 : ab = 0\}$. So W is the union of the two coordinate axes in $\mathbb{R}^2$. Each of these coordinate axes is a subspace of $\mathbb{R}^2$, but W is not. For instance, $(1, 0), (0, 1) \in W$, but $(1, 0) + (0, 1) = (1, 1) \notin W$. So W is not closed under addition.

4. $\{0\}$ and V are always subspaces of V. The empty set $\emptyset$ is not a subspace (since it does not contain 0).

Proposition. If W_1 and W_2 are subspaces of V, so is $W_1 \cap W_2$.

Proof. Since W_1 and W_2 are subspaces, we have $0 \in W_i$ for $i = 1, 2$. Hence, $0 \in W_1 \cap W_2$. If $u, v \in W_1 \cap W_2$, then $u, v \in W_i$ for $i = 1, 2$. Hence, $u + v \in W_i$ for $i = 1, 2$. Similarly, for each $\lambda \in F$,

$$\begin{aligned} u \in W_1 \cap W_2 \quad &\Rightarrow \quad u \in W_1 \text{ and } u \in W_2 \\ &\Rightarrow \quad \lambda u \in W_1 \text{ and } \lambda u \in W_2 \\ &\Rightarrow \quad \lambda u \in W_1 \cap W_2. \end{aligned}$$

□

Proposition. Let S be a subset of V. Then:

1. $\operatorname{Span}(S)$ is a subspace of V.
2. If $W \subseteq V$ is a subspace and $S \subseteq W$, then $\operatorname{Span}(S) \subseteq W$. (In other words: a subspace is closed under the process of taking linear combinations of its elements.)
3. Every subspace of V is the span of some subset of V.

Proof. 1. If $S = \emptyset$, then $\operatorname{Span}(S) = \{0\}$, which is a subspace of V. Otherwise, we will show $0 \in \operatorname{Span}(S)$ and $\operatorname{Span}(S)$ is closed under addition and scalar multiplication. Since $S \neq \emptyset$, there exists some $u \in S$. Then $0 \cdot u$ is a linear combination of elements in S, and $0 \cdot u = 0$ (the first 0 in this equation is in F, and the second is in V). Hence, $0 \in \operatorname{Span}(S)$. Now let $x, y \in \operatorname{Span}(S)$ so that

$$\begin{aligned} x &= a_1 u_1 + \cdots + a_m u_m \\ y &= b_1 v_1 + \cdots + b_n v_n \end{aligned}$$

for some $a_i, b_i \in F$ and $u_i, v_i \in S$. Then

$$x + y = a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n \in \mathrm{Span}(S)$$

and for each $\lambda \in F$,

$$\lambda x = \lambda(a_1u_1 + \cdots + a_mu_m) = (\lambda a_1)u_1 + \cdots + (\lambda_m a_m)u_m \in \mathrm{Span}(S).$$

2. Take $x \in \mathrm{Span}(S)$. Then $x = a_1u_1 + \cdots + a_mu_m$ for some $a_i \in F$ and $u_i \in S$. Since $S \subseteq W$, we have $u_i \in W$ for all i, and since W is a subspace, it is closed under vector addition and scalar multiplication. Therefore, $x \in W$.

3. $\mathrm{Span}(W) = W$. □

Definition. A subset $S \subseteq V$ *generates* a subspace W if $\mathrm{Span}(S) = W$.

Examples.

1. $\{1, x, x^2, \ldots, \}$ generates $P(F)$, the vector space of polynomials in one variable over F. More commonly, this vector space is denoted $F[x]$.

2. $\{(1, 0), (0, 1)\}$ generates $\mathbb{R}^2$. So do $\{(1, 0), (0, 1), (3, -2)\}$ and $\{(1, 1), (0, 1)\}$.

3. The *i-the standard basis vector* for F^n is $e_i := (0, \ldots, 0, 1, 0, \ldots, 0)$, the vector whose only nonzero entry is in the i-th component. We have that $\{e_1, \ldots, e_n\}$ generates F^n.