Math 201 lecture for Wednesday, Week 1

Reduced row echelon form

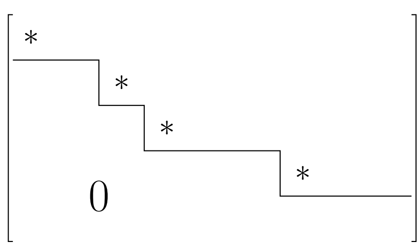
Goals for today:

- Discuss the computation of the reduced row echelon form of a system (or matrix).
- If there is an infinite number of solutions to a system, know how to describe the solution set in two ways (which we will call "parametric form" and "vector form").
- Vocabulary: reduced row echelon form, pivot variables (or pivot columns or pivot entries), free variables.

Procedure:

1. Convert the linear system into an augmented matrix.
2. Compute the reduced row echelon form of the matrix.
3. Convert the reduced matrix back into a system of equations.
4. Solve for the pivot variables.
5. Express your solution in one of two forms, as described in the examples below.

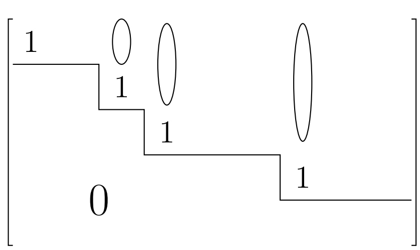
Echelon forms See our text for the precise definitions of *echelon form* (Chapter I, Definition 1.10) and *reduced row echelon form* (Chapter III, Definition 1.3). The general structure of a matrix in row echelon form is:



$*$ = nonzero *pivot* entry

zeros below "staircase"

The general structure of a matrix in *reduced* row echelon form is:



pivot entries = 1

zeros below "staircase"
and above pivot entries

Example 1. Find the solutions to the following linear system over the real numbers:

$$\begin{aligned} 2x_3 + 6x_4 \quad &= 0 \\ x_1 + 2x_2 + \ \ x_3 + 3x_4 \quad &= 1 \\ 2x_1 + 4x_2 + 3x_3 + 9x_4 + x_5 &= 5. \end{aligned}$$

Solution: The associated augmented matrix is

$$\left(\begin{array}{ccccc|c} 0 & 0 & 2 & 6 & 0 & 0 \\ 1 & 2 & 1 & 3 & 0 & 1 \\ 2 & 4 & 3 & 9 & 1 & 5 \end{array}\right).$$

We now use row operations to compute the reduced row echelon form of this matrix:

$$\left(\begin{array}{ccccc|c} 0 & 0 & 2 & 6 & 0 & 0 \\ 1 & 2 & 1 & 3 & 0 & 1 \\ 2 & 4 & 3 & 9 & 1 & 5 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 0 & 1 \\ 0 & 0 & 2 & 6 & 0 & 0 \\ 2 & 4 & 3 & 9 & 1 & 5 \end{array}\right) \xrightarrow{r_3 \to r_3 - 2r_1}$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 0 & 1 \\ 0 & 0 & 2 & 6 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 3 \end{array}\right) \xrightarrow{r_2 \to \frac{1}{2} r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 3 \end{array}\right) \xrightarrow[r_3 \to r_3 - r_2]{r_1 \to r_1 - r_2}$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right).$$

The final matrix is the *reduced row echelon form* of the original matrix. (See our text for the precise definition.) The first, third, and fifth column are the *pivot columns*, and the leading 1s in these column are called *pivots*.

The system of linear equations represented by the reduced echelon form has the same set of solutions as the original system and is in a much simpler form:

$$\begin{aligned} x_1 + 2x_2 \qquad\qquad &= 1 \\ x_3 + 3x_4 \quad &= 0 \\ x_5 &= 3. \end{aligned}$$

The variables corresponding to the pivots—in this case, x_1, x_3, and x_5—are called the *pivot variables*. The others—in this case, x_2 and x_4—are the *free variables*. The free variables can take on any values, and once we assign values to the free variables, they determine the values of the pivot variables. (Since we have two free variables, the solution set is *two-dimensional* in a sense that will be precisely defined later in the course.) The solution set is

$$\left\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 + 2x_2 = 1, x_3 + 3x_4 = 0, x_5 = 3\right\}.$$

Solving for the pivot variables, we have

$$\begin{aligned} x_1 &= 1 - 2x_2 \\ x_3 &= -3x_4 \\ x_5 &= 3. \end{aligned}$$

You should know how to express this solution in the following two ways. The first involves writing the pivot variables in terms of the free variables:

$$\{(1 - 2x_2, x_2, -3x_4, x_4, 3) : x_2, x_4 \in \mathbb{R}\}.$$

We will call this the *parametric* version of the solution set. The second way of writing the solution set we will call the *vector* version. It looks like this:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} : x_2, x_4 \in \mathbb{R} \right\}.$$

Here we are using *column vectors*, and for instance,

$$x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and we can add column vectors component-wise. Therefore,

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 - 2x_2 \\ x_2 \\ -3x_4 \\ x_4 \\ 3 \end{pmatrix}.$$

Example 2. Suppose the reduced row echelon form for an augmented matrix has the form

$$\left(\begin{array}{ccccccc|c} 0 & 1 & 0 & 1 & 2 & 0 & 1 & 7 \\ 0 & 0 & 1 & 4 & -1 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Write the solution set in parametric and vector form.

Solution: The first step is to convert the augmented matrix as a system of equations:

$$\begin{aligned} x_2 \quad + \quad x_4 + 2x_5 \quad + \quad x_7 &= 7 \\ x_3 + 4x_4 - \quad x_5 \quad + 3x_7 &= -2 \\ x_6 + \quad x_7 &= 3. \end{aligned}$$

(The last row corresponds to the equation $0 = 0$, which we can discard.) The pivot variables are x_2, x_3, and x_6. The rest are free variables. Solving for the pivot variables:

$$\begin{aligned} x_2 &= 7 - x_4 - 2x_5 - x_7 \\ x_3 &= -2 - 4x_4 + x_5 - 3x_7 \\ x_6 &= 3 - x_7. \end{aligned}$$

The parametric form for the solutions is

$$\{(x_1,\ 7 - x_4 - 2x_5 - x_7,\ -2 - 4x_4 + x_5 - 3x_7,\ x_4,\ x_5,\ 3 - x_7,\ x_7) : x_4, x_5, x_7 \in \mathbb{R}\}, \tag{1}$$

and the vector form is

$$\left\{ \begin{pmatrix} 0 \\ 7 \\ -2 \\ 0 \\ 0 \\ 3 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ -1 \\ -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_7 \begin{pmatrix} 0 \\ -1 \\ -3 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} : x_4, x_5, x_7 \in \mathbb{R} \right\}.$$

Important: Note how these two forms of the solutions are related. For instance, looking at just the constant terms in (1) gives the first column vector displayed above. Looking at just the coefficients of x_4 gives the second column vector, and so on. We get one column for the constants and one column for each of the free variables.

The next example illustrates how linear algebra can be used to say something about non-linear objects.

Example 3. Find all parabolas $f(x) = ax^2 + bx + c$ passing through the points $(1, 4)$ and $(3, 6)$ (determine a, b, and c).

Solution: To pass through $(1, 4)$, we need $f(1) = 4$, i.e.,

$$4 = a \cdot 1^2 + b \cdot 1 + c = a + b + c,$$

and to pass through $(3, 6)$, we need $f(3) = 6$, i.e.,

$$6 = a \cdot 3^2 + b \cdot 3 + c = 9a + 3b + c.$$

So we need to solve the system

$$\begin{aligned} a + b + c &= 1 \\ 9a + 3b + c &= 6. \end{aligned}$$

Apply our algorithm:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 9 & 3 & 1 & 6 \end{array}\right) \xrightarrow{r_2 \to r_2 - 9r_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -6 & -8 & -30 \end{array}\right) \xrightarrow{r_2 \to -\frac{1}{6} r_2}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 4/3 & 5 \end{array}\right) \xrightarrow{r_1 \to r_1 - r_2} \left(\begin{array}{ccc|c} 1 & 0 & -1/3 & -1 \\ 0 & 1 & 4/3 & 5 \end{array}\right).$$

The corresponding system is

$$\begin{aligned} a \quad - \tfrac{1}{3}c &= -1 \\ b + \tfrac{4}{3}c &= 5. \end{aligned}$$

The solution set is

$$\left\{\left(-1 + \tfrac{1}{3}c, 5 - \tfrac{4}{3}c, c\right) : c \in \mathbb{R}\right\}$$

or

$$\left\{ \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1/3 \\ -4/3 \\ 1 \end{pmatrix} : c \in \mathbb{R} \right\}.$$

In this way, the set of parabolas passing through the two given points is parametrized by a line in 3-space. For each c, we get a corresponding parabola—the graph of the function

$$f(x) = \left(-1 + \tfrac{1}{3}c\right) x^2 + \left(5 - \tfrac{4}{3}c\right) x + c.$$

Here is a check that all of these parabolas pass through $(1, 4)$ and $(3, 6)$:

$$\begin{aligned} f(1) &= \left(-1 + \tfrac{1}{3}c\right) + \left(5 - \tfrac{4}{3}c\right) + c = 4 \\ f(3) &= \left(-1 + \tfrac{1}{3}c\right) 3^2 + \left(5 - \tfrac{4}{3}c\right) 3 + c = 6. \end{aligned}$$

Here are graphs of a few of these ($c = -1, 0, 1, 2, 3, 4, 5$):