Math 201 lecture for Friday, Week 1

Introduction to $\mathbb{R}$

We will start the study of abstract vector spaces in the next class. Today, we will introduce a particular vector space, $\mathbb{R}^n$, and informally discuss some of its subspaces (lines, planes, etc.)

Definition. *Real n-space* is the set

$$\mathbb{R}^n := \underbrace{\mathbb{R}\times\cdots\times\mathbb{R}}_{n\text{-factors}} := \{(x_1,\ldots,x_n) : x_i \in \mathbb{R} \text{ for } i = 1,\ldots,n\}.$$

with two operations: *addition* $+\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and *scalar multiplication* $\cdot\colon \mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ defined, respectively, as follows:

$$(x_1,\ldots,x_n)+(y_1,\ldots,y_n) := (x_1+y_1,\ldots,x_n+y_n),$$

and

$$\lambda(x_1,\ldots,x_n) := (\lambda x_1,\ldots,\lambda x_n)$$

for all $(x_1,\ldots,x_n),(y_1,\ldots,y_n)\in\mathbb{R}^n$ and $\lambda\in\mathbb{R}$.

Example 1. In $\mathbb{R}^4$,

$$(4,0,-2,1)+(3,1,2,-4) = (7,1,0,-3)$$

and

$$2(0,3,3,7) = (0,6,6,14).$$

We will often think of points in $(x_1,\ldots,x_n)\in\mathbb{R}^n$ as column vectors or column matrices:

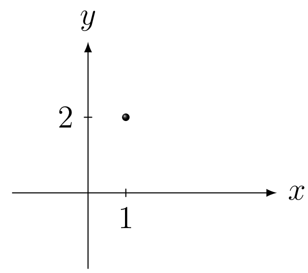
$$\begin{pmatrix} x_1\\ \vdots \\ x_n\end{pmatrix}.$$

Example 2.

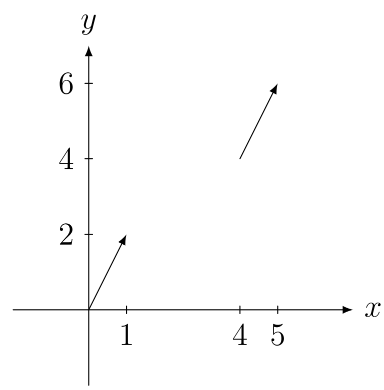
$$2\begin{pmatrix}1\\2\\4\end{pmatrix} - \begin{pmatrix}3\\0\\-4\end{pmatrix} = \begin{pmatrix}-1\\4\\12\end{pmatrix}.$$

There are two standard ways to interpret an element $p\in\mathbb{R}^n$: either as *point* in space or as a *vector* (thought of as an arrow or direction[1]) For instance, we can think of $(1,2)\in\mathbb{R}^2$ as point:

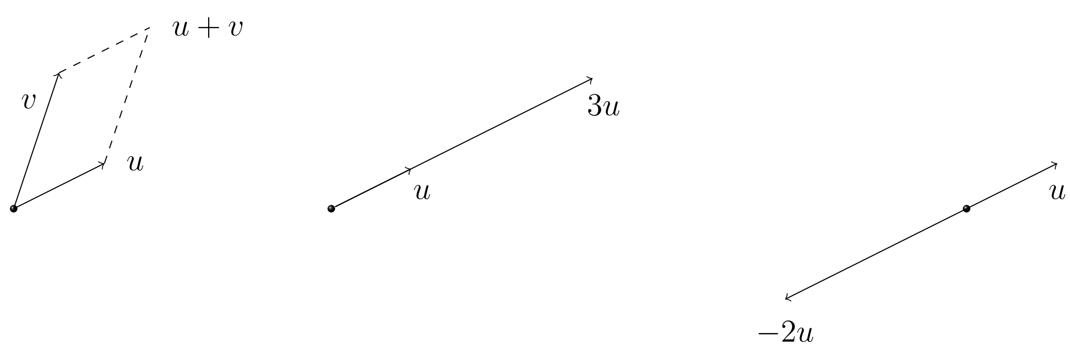
[1] The word "vector" will soon have a technical definition: an element of a vector space.



or as an arrow/direction:



Geometrically, addition of vectors is given by the “parallelogram rule” and scalar multiplication amounts to scaling the length of a vector but not its directions (except that scaling by a negative number reverses direction):



Definition (line in $\mathbb{R}^n$). Let $p, v \in \mathbb{R}^n$, with $v \neq (0, \ldots, 0)$. Then

$$\{p + \lambda v : \lambda \in \mathbb{R}\}$$

is the *line in $\mathbb{R}^n$ in the direction of v and passing through the point p.*

Remarks.

1. In the above definition, the set of all scalar multiples of v, i.e., $\{\lambda v : \lambda \in \mathbb{R}\}$, gives a line *through the origin* in the direction of v. We then translate that line through the origin by p. In that translation, the origin is translated to the point p (and, thus, the resulting line contains the point p).

2. We say the function

$$\begin{aligned} \mathbb{R} &\to \mathbb{R}^n \\ t &\mapsto p + tv \end{aligned}$$

is a *parametrization* of the line passing through p in the direction of v. Its image is the line, itself. We could use λ or any symbol here instead of t, of course. We are using t to connote time. We think of the parametrization as giving the position at each time t of a point traveling along the line.)

3. **Exercise.** Try to show that if q is any point on the line $\{p + \lambda v : \lambda \in \mathbb{R}\}$ and w is any nonzero scalar multiple of v, then

$$\{q + \lambda w : \lambda \in \mathbb{R}\} = \{p + \lambda v : \lambda \in \mathbb{R}\}$$

So the same line may be described in many different ways. (This is not hard to do: start with an arbitrary element of the set on the left-hand side, and then show that is contained in the set on the right-hand side.)

Example 3. Give a parametrization of the line through the points $(1, 2, 0)$ and $(0, 1, 1)$ in $\mathbb{R}^3$.

Solution. For the direction, we may choose $v = (0, 1, 1) - (1, 2, 0) = (-1, -1, 1)$. (We think of v as the vector with head $(0, 1, 1)$ and tail $(1, 2, 0)$). We then pick any point on the line, say $p = (1, 2, 0)$. Then our line has the parametrization

$$t \mapsto p + tv = (1, 2, 0) + t\,((0, 1, 1) - (1, 2, 0)) = (1, 2, 0) + t(-1, -1, 1).$$

Setting t equal to 0 and 1, respectively, shows that this line really does pass throught the points $(1, 2, 0)$ and $(0, 1, 1)$. Here is an alternative way to write this parametrization:

$$t \mapsto (1 - t, 2 - t, t).$$

Example 4. *A line in $\mathbb{R}^2$ can always be expressed as the solution set to a single linear equation.* For example, consider the line

$$L := \{(3, 2) + \lambda(1, 5) : \lambda \in \mathbb{R}\}.$$

A point (x, y) lies on this line L if and only if

$$(x, y) = (3, 2) + \lambda(1, 5) = (3 + \lambda, 2 + 5\lambda)$$

for some $\lambda \in \mathbb{R}$. It follows that

$$\begin{aligned}
(x, y) \in L &\iff x = 3 + \lambda \quad \text{and} \quad y = 2 + 5\lambda \text{ for some } \lambda \\
&\iff x - 3 = \lambda \quad \text{and} \quad \frac{1}{5}(y - 2) = \lambda \text{ for some } \lambda \\
&\iff x - 3 = \frac{1}{5}(y - 2) \\
&\iff 5x - 15 = (y - 2) \\
&\iff 5x - y = 13.
\end{aligned}$$

So the line L is the set of solutions to $5x - y = 13$.

Example 5. *A line in $\mathbb{R}^3$ is always the solution set to a system of two linear equations.* For example, consider the line through the points $(1, 0, 2)$ and $(3, 1, -1)$, parametrized by

$$t \mapsto (1, 0, 2) + t\left((3, 1, -1) - (1, 0, 2)\right) = (1, 0, 2) + t(2, 1, -3).$$

We would like to find a system of two linear equations whose solution set is this line. A linear equation in $\mathbb{R}^3$ has the form

$$ax + by + cz = d.$$

We would like to find a, b, c, d so that the solution set for this equation contains the points $(1, 0, 2)$ and $(3, 1, -1)$. It turns out that this will force the whole line to be contained in the solution set. To contain these points, we need

$$\begin{aligned}
a \phantom{{}+b} + 2c &= d \\
3a + b - c &= d.
\end{aligned}$$

To solve this system, we first put it in reduced row echelon form by substracting 3 times the first equation from the second to get

$$\begin{aligned}
a \phantom{{}+b} + 2c &= d \\
b - 7c &= -2d.
\end{aligned}$$

Solve for the pivot variables:

$$\begin{aligned}
a &= -2c + d \\
b &= 7c - 2d.
\end{aligned}$$

To find two *independent* solutions[2], we first set $(c, d) = (1, 0)$, in which case

$$a = -2, b = 7, c = 1 \text{ and } d = 0.$$

We stick these values into $ax + by + cz = d$ to get our first linear equation

$$-2x + 7y + z = 0.$$

Next we set $(c, d) = (0, 1)$ to get

$$a = 1, b = -2, c = 0 \text{ and } d = 1$$

with corresponding linear equation

$$x - 2y = 1.$$

In sum, our line is the solution set to the system

$$\begin{aligned} -2x + 7y + z &= 0 \\ x - 2y \quad\;\; &= 1. \end{aligned}$$

The reader should check that our original points, $(1, 0, 2)$ and $(3, 1, -1)$ are both solutions to this system.

Definition (plane in $\mathbb{R}^n$). Let $p, v, w \in \mathbb{R}^n$. Suppose that v and w are nonzero and that neither is a scalar multiple of the other. Then

$$\left\{p + \lambda v + \mu w : (\lambda, \mu) \in \mathbb{R}^2\right\}$$

is the plane in $\mathbb{R}^n$ containing p and with directions[3] v and w.

The plane in the above definition has the *parametrization*

$$\begin{aligned} \mathbb{R}^2 &\to \mathbb{R}^n \\ (s, t) &\mapsto p + sv + tw. \end{aligned}$$

Example 6. Find the plane P through the points $(0, 2, -1)$, $(4, 2, 1)$, and $(1, 0, 1)$. Describe both parametrically and as the solution set to a single linear equation.

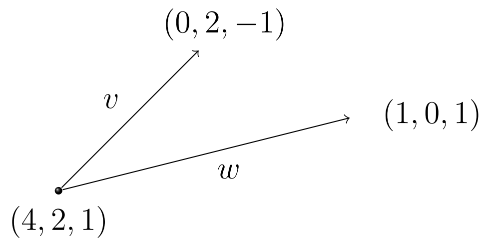
Solution. To find the directions, we fix any of the three points, say $(4, 2, 1)$, and we consider the two arrows having this point as their tail and $(0, 2, -1)$ and $(1, 0, 1)$ as their heads:

$$\begin{aligned} v &= ((0, 2, -1) - (4, 2, 1)) = (-4, 0, -2) \\ w &= ((1, 0, 1) - (4, 2, 1)) = (-3, -2, 0). \end{aligned}$$

[2]The precise meaning of *independence* is left for later.

[3]The word "directions" here is not quite standard but will do for now

Geometrically, we have the following picture for these directions:



So a parametric description of the line is

$$\{(4,2,1)+\lambda(-4,0,2)+\mu(-3,-2,0) : \lambda,\mu \in \mathbb{R}\}\,.$$

or

$$\{(4-4\lambda-3\mu, 2-2\mu, 1+2\lambda) : \lambda,\mu \in \mathbb{R}\}\,.$$

As an aside: this parametric description is great for drawing the plane using a computer. For instance, the Sage code for plotting this plane could be:

```
s,t = var("s,t")
parametric_plot3d((4-4*s-3*t,2-2*t,1+2*s),(s,-1,1),(t,-1,1))
```

We would now like to describe the plane P as the solution set of a single linear equation. That equation will have the form

$$ax+by+cz=d$$

for some $a,b,c,d \in \mathbb{R}$. Our job is to find a,b,c,d. (Scaling an equation does not change its solution set, so our solution will only be unique up to such scaling.) For P to contain $(0,2,-1)$, $(4,2,1)$, and $(1,0,1)$, we need

$$\begin{aligned} 2b - c &= d \\ 4a + 2b + c &= d \\ a \qquad + c &= d. \end{aligned}$$

Reducing to row echelon form and solving for pivot variables, we find

$$\begin{aligned} a &= -d \\ b &= \frac{3}{2}d \\ c &= 2d. \end{aligned}$$

We are free to choose any nonzero value for d (again: our solution is only unique up to scaling). We choose $d = 2$ (to get rid of the denominator of 2). Therefore, the plane P is the solution set to the equation

$$-2x + 3y + 4z = 2.$$