Math 201 Homework for Tuesday, Week 13

Due: Tuesday, November 30.

PROBLEM 1. Consider the inner product space $(\mathbb{R}^2, \langle\ ,\ \rangle)$ presented as an example in class, where the inner product is defined as

$$\langle (x_1, x_2), (y_1, y_2) \rangle = 3x_1y_1 + 2x_1y_2 + 2x_2y_1 + 4x_2y_2.$$

(a) Compute the lengths of the vectors $(1, 1)$ and $(1, -1)$.

(b) Compute the cosine of the angle between $(1, 1)$ and $(1, -1)$. Are these vectors perpendicular?

(c) Find a non-zero vector perpendicular to $(1, 1)$.

PROBLEM 2. Recall the inner product defined on $M_{m\times n}(F)$, where $F = \mathbb{R}$ or $\mathbb{C}$: for $A, B \in M_{m\times n}(F)$, we define

$$\langle A, B \rangle = \operatorname{tr}(B^* A),$$

where $B^* = \overline{B^t}$ is the conjugate transpose. In this problem we will verify that this function does indeed satisfy the axioms of an inner product.

(a) Prove that this function is linear in the first coordinate: for all $A, B, C \in M_{m\times n}(F)$ and $r \in F$,

$$\langle A + rC, B \rangle = \langle A, B \rangle + r\langle C, B \rangle.$$

(b) Prove that this function is conjugate symmetric: for all $A, B \in M_{m\times n}(F)$,

$$\langle A, B \rangle = \overline{\langle B, A \rangle}.$$

(c) Prove that this function is positive-definite: for all $A \in M_{m\times n}(F)$ with $A \neq 0$,

$$\langle A, A \rangle > 0.$$

PROBLEM 3. Let $(V, \langle\ ,\ \rangle)$ be an inner product space over $\mathbb{R}$, and let $v, w \in V$ be nonzero vectors.

(a) Prove that if the vector

$$\frac{v}{\|v\|} + \frac{w}{\|w\|}$$

is nonzero, then it bisects the angle between v and w.

(b) Illustrate in $\mathbb{R}^2$ with the standard dot product.