Math 201 Homework for Friday, Week 13

Due: Friday, December 3.

PROBLEM 1. Let $S = \{(1, 0, i), (1, 2, 1)\}$ in $\mathbb{C}^3$ (with the standard inner product). Compute $S^{\perp}$.

PROBLEM 2. Let $V = \mathcal{P}(\mathbb{R})$ be the vector space of all polynomials with coefficients in $\mathbb{R}$, with inner product $\langle f(x), g(x)\rangle = \int_0^1 f(t)g(t)\,dt$. Let $W = \mathcal{P}_1(\mathbb{R})$. (Warning: to get this problem right, you will need to be very careful with your calculations and double-check your solutions.)

(a) Find an orthonormal basis $\{u_1, u_2\}$ for W.

(b) Find the closest polynomial in W to $h(x) = 3 - 2x + x^2$. Express your solution in two forms: (i) as a linear combination of u_1 and u_2, and (ii) as a linear combination of 1 and x.

PROBLEM 3. Let V be an n-dimensional vector space over $F = \mathbb{R}$ or $\mathbb{C}$, and let $\langle\, ,\,\rangle$ be an inner product on V. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be an ordered basis for V (not necessarily orthonormal). Let A be the $n \times n$ matrix given by

$$A_{ij} = \langle v_j, v_i\rangle.$$

Recall that for $x \in V$, $[x]_{\mathcal{B}}$ denotes the coordinate vector for x with respect to the basis $\mathcal{B}$, and as usual, we will think of this vector in F^n as an $n \times 1$ matrix.

(a) Prove that for all $x, y \in V$,

$$\langle x, y\rangle = ([y]_{\mathcal{B}})^* \, A \, ([x]_{\mathcal{B}})\,.$$

(Recall that for a matrix C, we define $\overline{C}$ by $\overline{C}_{ij} = \overline{(C_{ij})}$, and then we define the conjugate transpose by $C^* = \overline{C^t}$. *Hint*: compute both sides using sum notation. On the right-hand side, you will be computing the $(1, 1)$-entry of a 1×1 matrix.)

(b) Prove that the matrix A satisfies $A = A^*$.

(c) If the basis $\mathcal{B}$ is orthonormal, what is the matrix A?

(d) (Bonus) Let $\mathcal{D}$ be another ordered basis for V, and let C be the associated $n \times n$ matrix. How are A and C related?