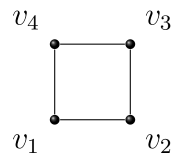
Math 201 Homework for Tuesday, Week 12

Due: Tuesday, November 23.

PROBLEM 1. Consider the *cycle graph* C_4:



(a) Find the adjacency matrix $A = A(G)$.

(b) Compute A^4 and use it to determine the number of walks from v_1 to v_3 of length 4. List all of these walks (these will be ordered lists of 5 vertices).

(c) What is the total number of *closed* walks of length 4?

(d) Compute and factor the characteristic polynomial for A.

(e) What are the algebraic multiplicities of each of the eigenvalues?

(f) Diagonalize A using our algorithm: compute bases for the eigenspaces of each of the eigenvalues you just found, and use them to construct a matrix P such that $P^{-1}AP$ is a diagonal matrix with the eigenvalues along the diagonal.

(g) Use part (f) to find a closed expression for A^ℓ for each $\ell \geq 1$.

(h) Take the trace of A^ℓ to get a formula for the number of closed walks of length ℓ for each $\ell \geq 1$. (You can check your result against the formula given in class.)

PROBLEM 2. In this exercise we will prove the theorem from class:
"Let A be the adjacency matrix for a graph G with vertices $v_1, \ldots, v_n$, and let $\ell \in \mathbb{Z}_{\geq 0}$. Then then number of walks of length ℓ from v_i to v_j is $(A^\ell)_{ij}$."

(a) Let $p(i,j,\ell)$ denote the number of walks of length ℓ in G from v_i to v_j. Prove that for all $i,j = 1, \ldots, n$ and $\ell \geq 1$,

$$p(i,j,\ell) = \sum_{k=1}^{n} p(i,k,\ell-1)p(k,j,1).$$

(*Hint:* Part of the trick is to parse this formula appropriately.)

(b) Prove the theorem by induction on ℓ, using the result from part (a).