Math 201 Homework for Tuesday, Week 11

Due: Tuesday, November 16.

REMINDER: You have a presentation proposal due next Tuesday, November 23. Please start thinking about the topic of your presentation: an application of linear algebra to other fields.

PROBLEM 1. Let $V = \mathcal{P}_3(\mathbb{R})$, the vector space over $\mathbb{R}$ consisting of all polynomials with real coefficients having degree at most 3. Define the following linear transformation on V (in which the prime denotes differentiation),

$$\begin{aligned} L\colon V &\to V \\ f &\mapsto xf' + f'. \end{aligned}$$

(a) Write the matrix of L with respect to the ordered basis $\langle 1, x, x^2, x^3 \rangle$ of V.

(b) What are the eigenvalues of L?

(c) Does V have a basis of eigenvectors of L? If so, give such a basis (**written as polynomials** not tuples of real numbers), and if not, explain why not.

PROBLEM 2. Consider the matrix

$$B = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

(a) What are the algebraic and geometric multiplicities of each of the eigenvalues of B?

(b) Explain whether B is diagonalizable in terms of the geometric multiplicities of its eigenvalues.

PROBLEM 3. Consider an $n \times n$ matrix A over $\mathbb{C}$. As mentioned in class, the characteristic polynomial of A is of the form

$$p_A(t) = \det(A - tI_n) = (-1)^n t^n + b_{n-1}t^{n-1} + \cdots + b_1 t + b_0.$$

(a) Prove that b_0 is equal to the determinant of A.

(b) Prove that $b_{n-1} = (-1)^{n-1}\operatorname{tr}(A)$, where tr denotes the trace.

(HINT: Using the permutation expansion identify where the possible terms of degree $n-1$ arise from.)

(c) Prove that the product of all eigenvalues of A (with multiplicity) is equal to $\det(A)$. (You may use the fact the characteristic polynomial will factor completely into linear factors, $p_A(t) = (-1)^n \prod_{i=1}^{n}(x_i - \lambda_i)$, since we are working over $\mathbb{C}$.)

(d) Prove that the sum of all eigenvalues of A (with multiplicity) is equal to $\operatorname{tr}(A)$.

NOTE: When $n = 2$, this result says that

$$p_A(t) = t^2 - \operatorname{tr}(A)t + \det(A).$$