Math 201 Homework for Friday, Week 11

Due: Friday, November 19.

PROBLEM 1. Consider a sequence of numbers p_n defined recursively by fixing constants a and b, next assigning initial values for p_0 and p_1, and then for $n \geq 1$ letting

$$p_{n+1} = ap_n + bp_{n-1}.$$

For instance, letting $a = 2$, $b = -1$, $p_0 = 0$, and $p_1 = 1$, we get

$$\begin{aligned} p_0 &= 0 \\ p_1 &= 1 \\ p_{n+1} &= 2p_n - p_{n-1} \quad \text{for } n \geq 1, \end{aligned}$$

which defines the sequence

$$0, 1, 2, 3, 4, 5, 6, \ldots$$

Given any sequence of this form, we can encode the recursive relation in the following matrix equation:

$$\begin{pmatrix} p_{n+1} \\ p_n \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_n \\ p_{n-1} \end{pmatrix}. \tag{1}$$

So we have

$$\begin{pmatrix} p_2 \\ p_1 \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_0 \end{pmatrix},$$

which implies

$$\begin{pmatrix} p_3 \\ p_2 \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_2 \\ p_1 \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \left[\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_0 \end{pmatrix} \right] = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} p_1 \\ p_0 \end{pmatrix},$$

and so on. In general, we have

$$\begin{pmatrix} p_{n+1} \\ p_n \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} p_1 \\ p_0 \end{pmatrix}. \tag{2}$$

Let

$$A = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}.$$

and suppose A is diagonalizable. Take P so that

$$P^{-1}AP = D = \mathrm{diag}(\lambda_1, \lambda_2).$$

We have seen that it follows that $A^n = PD^nP^{-1}$, so that equation (2) becomes

$$\begin{pmatrix} p_{n+1} \\ p_n \end{pmatrix} = PD^nP^{-1}\begin{pmatrix} p_1 \\ p_0 \end{pmatrix} = P\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}P^{-1}\begin{pmatrix} p_1 \\ p_0 \end{pmatrix}.$$

Thus, we get a closed form expression for p_n in terms of powers of the eigenvalues of A (just take the second component of the product on the right-hand side of the above equation).

Let $a = b = 1$, $p_0 = 0$, and $p_1 = 1$.

(a) Write out the first several values for the sequence (p_n).

(b) Write the corresponding matrix equation, as (1) above.

(c) Diagonalize the matrix A and compute the corresponding equation for p_n in terms of powers of the eigenvalues of A.

You may find the following notation useful:

$$\phi = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \overline{\phi} = \frac{1-\sqrt{5}}{2},$$

with useful relations $\phi\overline{\phi} = -1$, $\phi^2 = \phi + 1$, and $\phi + \overline{\phi} = 1$. (**Warning:** You will want to make sure you get the diagonalization perfect. This will take some time. Using the above notation as much as possible will help.)

(d) As in the first example above, let $a = 2$, $b = -1$, $p_0 = 0$ and $p_1 = 1$. What happens when you try to use the method above to find a closed formula for p_n?

Problem 2. One of the reasons we like diagonalization is because computing the powers of the matrix is easy if it is diagonalized (see previous exercise). In this exercise we explore the powers of Jordan blocks. Recall that a 2×2 Jordan block is a matrix of the form

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

for some $\lambda \in F$.

(a) For J as above, compute J^2, J^3 and J^4. For a natural number n, make a conjecture for the value of J^n.

(b) Prove your conjecture. (*Hint:* You probably want to use induction.)

(c) (BONUS:) Repeat (a) and (b) above for a 3×3 Jordan block:

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$