Math 201 Homework for Tuesday, Week 10

Due: Tuesday, November 9.

PROBLEM 1. Compute the determinants of the following matrices by using the permutation expansion.

(a) $\begin{pmatrix} -1 & 2+i & 3 \\ 1-i & i & 1 \\ 3i & 2 & -1+i \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 4 & 0 \\ 5 & 0 & 0 & 0 & 0 \end{pmatrix}$

PROBLEM 2. Compute the determinants of the same matrices as in Problem 1 by using the Laplace expansion along any row or column. Be clear about which row or column you are using.

PROBLEM 3. Let $p(x_1, \ldots, x_n)$ be a polynomial on n variables with coefficients in a field F. An arbitrary term of this polynomial is of the form $ax_1^{d_1}x_2^{d_2}\ldots x_n^{d_n}$, where $a \in F$ and d_i is a nonnegative integer for all i. The total degree of this term is $d_1 + \cdots + d_n$.

For example, the polynomial

$$p(x_1, x_2, x_3) = 2 + x_1 + 3x_1x_2^2 - 4x_2^2x_3^3 + 9x_1x_2x_3$$

has five terms of total degree 0, 1, 3, 5, and 3, respectively.

Here is a result from polynomial algebra. If p satisfies the conditions:

(i) $p(x_1, \ldots, x_n) = 0$ whenever $x_i = x_j$ for $i \neq j$;

(ii) the total degree of every term is $n(n-1)/2$,

then

$$p(x_1, \ldots, x_n) = k(x_2 - x_1)(x_3 - x_1) \cdots (x_n - x_{n-1})$$

for some $k \in F$. Here the product contains all the terms of the form $x_j - x_i$ with $1 \leq i < j \leq n$. Note that the coefficient k is equal to the coefficient of $x_2x_3^2 \cdots x_{n-1}^{n-2}x_n^{n-1}$ in p.

For example, when $n = 2$ and $F = \mathbb{R}$, $p(x_1, x_2) = x_1 - x_2$ satisfies both properties, $p(x_1, x_2) = x_1^2 - x_2^2$ satisfies (i) but not (ii), and $p(x_1, x_2) = x_1 + 2x_2$ satisfies (ii) but not (i).

Now consider the Vandermonde matrix

$$V(x_1, \ldots, x_n) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}$$

Let $p(x_1, \ldots, x_n) = \det(V(x_1, \ldots, x_n))$.

(a) Using properties of determinants, prove that p satisfies property (i).

(b) Using the permutation expansion of the determinant, prove that p satisfies property (ii).

(*Hint:* As always, it is useful to play around with small cases of n to understand what is really going on. Try $n = 2$ and $n = 3$, and then use what you learn to argue for the general case. When you write your solutions, you should write them for arbitrary n.)

(c) It follows from (a) and (b) and the discussion above that

$$p(x_1, \ldots, x_n) = k(x_2 - x_1)(x_3 - x_1) \cdots (x_n - x_{n-1})$$

for some $k \in F$. Find the value of the coefficient k.

(d) (BONUS.) A general polynomial of degree d in one variable over the real numbers has the form

$$q(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_d x^d,$$

where the a_i are real numbers. Pick n distinct real numbers $x_1, \ldots, x_n$, and pick arbitrary (not necessarily distinct) real numbers $b_1, \ldots, b_n$. Prove that there is a unique polynomial $q(x)$ of degree $n-1$ over the real numbers such that $q(x_i) = b_i$ for $i = 1, \ldots, n$.

(e) (BONUS.) Use the Vandermonde determinant to prove that the collection of functions $\{e^{\alpha x} : \alpha \in \mathbb{R}\}$ is linearly independent. (Recall that the set of functions from $\mathbb{R}$ to $\mathbb{R}$ is a vector space. The solution to this problem would show that this space is infinite dimensional.)