Math 201 Homework for Friday, Week 9

Due: Friday, November 5.

PROBLEM 1. Let

$$A = \begin{pmatrix} 1 & -1 & 2 & -2 \\ -2 & 2 & 2 & 4 \end{pmatrix}.$$

Find elementary matrices $E_1, \ldots, E_\ell$ such that $E_\ell \cdots E_2 E_1 A$ is the reduced echelon form of $A$. (Check your work.)

PROBLEM 2. This exercise will prove that the determinant is multiplicative, that is, for $n \times n$ matrices $A$ and $B$,

$$\det(AB) = \det(A)\det(B).$$

Let $B$ be a fixed $n \times n$ matrix over $F$ such that $\det(B) \neq 0$. Consider the function

$$d\colon M_{n\times n}(F) \longrightarrow F$$

defined by $d(A) = \det(AB)/\det(B)$. You will prove that $d(A) = \det(A)$. For a matrix $A$, we write $(r_1, \ldots, r_n)$ for the rows of $A$, with each $r_i \in F^n$.

(a) Prove that $d$ is multilinear on rows, that is, $d$ satisfies that

$$d(r_1, \ldots, r_i + k \cdot r_i', \ldots, r_n) = d(r_1, \ldots, r_i, \ldots, r_n) + kd(r_1, \ldots, r_i', \ldots, r_n)$$

for all $r_1, \ldots, r_n, r_i' \in F^n$ and any $k \in F$.

(Some suggested notation to help in your proof: let $c_1, \ldots, c_n$ be the columns of $B$. Then $(AB)_{s,t} = r_s \cdot c_t$, i.e., the $s,t$-entry of $AB$ is the dot product of the $s$-th row of $A$ with the $t$-th column of $B$. Recall that the dot product is defined by $(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1 y_1 + \cdots + x_n y_n$. Letting $A'$ be the matrix with rows $(r_1, \ldots, r_i', \ldots, r_n)$ and $A''$ the matrix with rows $(r_1, \ldots, r_i + kr_i', \ldots, r_n)$, you will need compare the rows of $AB$, $A'B$ and $A''B$.)

(b) Prove that $d$ is alternating on rows, that is, $d$ satisfies that $d(r_1, \ldots, r_n) = 0$ if $r_i = r_j$ for some $i \neq j$.

(c) Prove that $d(I_n) = 1$.

(d) Deduce that for all $A$, we have that $d(A) = \det(A)$, and that $\det(AB) = \det(A)\det(B)$.

(e) Prove that $\det(AB) = \det(A)\det(B)$ when $\det(B) = 0$. (*Hint:* You have basically proved this in an earlier homework.)