Math 201 Homework for Friday, Week 8

Due: Friday, October 29.

PROBLEM 1. Compute the determinant of the following matrices by using row operations.

(a) $\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 3 & -1 & 2 \\ 2 & 4 & 7 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}$

(c) $\begin{pmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix}$

(d) (BONUS) Generalize part (c) for the $n \times n$ matrix $\begin{pmatrix} n & -1 & \cdots & -1 \\ -1 & n & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n \end{pmatrix}$, with n in the main diagonal and -1 everywhere else.

PROBLEM 2. Let V be a vector space. For each integer $r > 0$, we now give a provisional definition of a new vector space called $\bigwedge^r V$. A spanning set for $\bigwedge^r V$ consists of expressions of the form $v_1 \wedge \cdots \wedge v_r$ where $v_1, \ldots, v_r \in V$. For example, if $u, v, w \in V$, the following would be typical elements of $\bigwedge^2 V$:

$$\begin{aligned} \omega &= 4\,u \wedge v - 5\,u \wedge w + 7\,v \wedge w \\ \mu &= 2\,u \wedge v + 9\,u \wedge w + 6\,v \wedge w. \end{aligned}$$

Addition is done by combining like terms, and scaling is done by scaling each term. For instance, continuing the example above, we get

$$\begin{aligned} \omega + \mu &= 6\,u \wedge v + 4\,u \wedge w + 13\,v \wedge w \\ 5\,\omega &= 20\,u \wedge v - 25\,u \wedge w + 35\,v \wedge w. \end{aligned}$$

We now add a couple of rules. First, these "wedge products" of vectors are linear in each component. For $v_i \in V$ and $a \in F$,

$$v_1 \wedge \cdots \wedge v_{i-1} \wedge (av_i + v_i') \wedge v_{i+1} \wedge \dots v_r =$$

$$\begin{gathered} a\, v_1 \wedge \cdots \wedge v_{i-1} \wedge v_i \wedge v_{i+1} \wedge \cdots \wedge v_r \\ + \\ v_1 \wedge \cdots \wedge v_{i-1} \wedge v_i' \wedge v_{i+1} \wedge \cdots \wedge v_r. \end{gathered}$$

Second, we declare that $v_1 \wedge \cdots \wedge v_r = 0$ if $v_i = v_j$ for some $i \neq j$. To illustrate these rules in action suppose $u, v, w \in V$. Then in $\bigwedge^3 V$, we have the following:

$$\begin{aligned} u \wedge (2u + 3v + 5w) \wedge w &= u \wedge (2u) \wedge w + u \wedge (3v) \wedge w + u \wedge (5w) \wedge w \\ &= 2\, u \wedge u \wedge w + 3\, u \wedge v \wedge w + 5\, u \wedge w \wedge w \\ &= 0 + 3\, u \wedge v \wedge w + 0 \\ &= 3\, u \wedge v \wedge w. \end{aligned}$$

Another example, this time in $\bigwedge^2 V$:

$$\begin{aligned} (u + 2v) \wedge (u + 3v) &= u \wedge (u + 3v) + (2v) \wedge (u + 3v) \\ &= u \wedge u + u \wedge (3v) + (2v) \wedge u + (2v) \wedge (3v) \\ &= 0 + 3\, u \wedge v + 2\, v \wedge u + 6\, v \wedge v \\ &= 3\, u \wedge v + 2\, v \wedge u + 0 \\ &= 3\, u \wedge v + 2\, v \wedge u. \end{aligned}$$

It turns out there is a little more we can do to simplify this last example. By the second rule, we have $(u+v) \wedge (u+v) = 0$, since in this expression we have two copies of the same vector. But linearly expanding this expression, we get

$$\begin{aligned} 0 &= (u + v) \wedge (u + v) \\ &= u \wedge (u + v) + v \wedge (u + v) \\ &= u \wedge u + u \wedge v + v \wedge u + v \wedge v \\ &= 0 + u \wedge v + v \wedge u + 0 \\ &= u \wedge v + v \wedge u. \end{aligned}$$

Thus, $u \wedge v + v \wedge u = 0$. This means that

$$u \wedge v = -v \wedge u.$$

In fact, in a wedge product of vectors, swapping any two locations negates the expression. (The proof is similar to the one we just gave in the case of $r = 2$.) For instance,

$$u \wedge v \wedge w = -u \wedge w \wedge v = w \wedge u \wedge v = -w \wedge v \wedge u.$$

Continuing our example from above, we get

$$\begin{aligned}(u + 2v) \wedge (u + 3v) &= \dots \quad \text{(see earlier calculation)}\\ &= 3\, u \wedge v + 2\, v \wedge u\\ &= 3\, u \wedge v - 2\, u \wedge v\\ &= u \wedge v.\end{aligned}$$

Now for some problems:

(a) Let $V = \mathbb{R}^2$, and take two vectors $u = (a, b)$ and $v = (c, d)$ in $\mathbb{R}^2$. Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Writing u and v as linear combinations of e_1 and e_2, find the number k in terms of a, b, c, d such that

$$u \wedge v = k\, e_1 \wedge e_2,$$

in $\bigwedge^2 V$. What is the relation between k and $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$?

(b) Now let $V = \mathbb{R}^3$, and take vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in $\mathbb{R}^3$. Writing these vectors as linear combinations of the standard basis vectors $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$, find numbers p, q, r in terms of the u_i and the v_i such that

$$u \wedge v = p\, e_2 \wedge e_3 - q\, e_1 \wedge e_3 + r\, e_1 \wedge e_2.$$

(Watch out for the minus sign in front of q.) Physics students may note a relation with the cross product of two vectors in $\mathbb{R}^3$.