Math 201 Homework for Tuesday, Week 7

Due: Tuesday, October 12.

Problem 1. Recall that if $P, Q \in M_{m\times n}(F)$, then their sum $P + Q \in M_{m\times n}(F)$ is defined to be the matrix with i, j-th entry

$$(P + Q)_{ij} := P_{ij} + Q_{ij},$$

On the other hand, if $P \in M_{m\times \ell}(F)$ and $Q \in M_{\ell\times n}(F)$ (note the change in dimensions), then their product $PQ \in M_{m\times n}(F)$ is defined to be the matrix whose i, j-th entry is

$$(PQ)_{ij} := \sum_{k=1}^{\ell} P_{ik}Q_{kj}.$$

Let $A \in M_{m\times \ell}(F)$, and let $B, C \in M_{\ell\times n}(F)$. Using only the definition of matrix addition and matrix multiplication, prove that

$$A(B + C) = AB + AC.$$

You do this by proving that the i, j-th entries on both sides are equal. **Please use summation notation, and be careful to specify the correct starting and ending points for the summation.** (The result you are proving is called the *left distributivity* property of matrix multiplication.)

Problem 2. For the following, recall that just as an example usually does not constitute a proof that something is true, a general discussion does not usually suffice to proof that something is not true. In the latter case, it is fine (but not necessary) to give a general discussion, but in the end, you should provide a concrete and simple counterexample.

(a) Prove that matrix multiplication of 2×2 matrices does not satisfy the commutative law, $AB = BA$.

(b) Prove that matrix multiplication of 2×2 matrices does not satisfy left cancellation.
Cancellation: If $AB = AC$ and A is not the zero matrix, then $B = C$.

Problem 3. Let $\mathcal{P}_n(\mathbb{R})$ be the vector space of polynomials in x of degree at most n with coefficients in $\mathbb{R}$. Let $f\colon \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ and $g\colon \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformations respectively defined as

$$f(p(x)) = (3 + x)p'(x) + 2p(x) \quad \text{and} \quad g(a + bx + cx^2) = (a + b, c, a - b).$$

Let $\mathcal{B} = \langle 1, x, x^2\rangle$ and $\mathcal{D}$ be the standard ordered basis for $\mathbb{R}^3$.

(a) Compute the matrix representing f with respect to the basis $\mathcal{B}$ for both the domain and codomain.

(b) Is f one-to-one? Is it onto?

(c) Compute the matrix representing g with respect to the bases $\mathcal{B}$ and $\mathcal{D}$.

(d) Compute the matrix representing $g \circ f$ with respect to the bases $\mathcal{B}$ and $\mathcal{D}$. Then use Theorem 2.7 (Chapter Three, Section IV) to verify your result. (This theorem says the composition of linear maps is represented by the matrix product of the representatives of the linear maps.)