Math 201 Homework for Friday, Week 7

Due: Friday, October 15.

Problem 1. The *trace* of an $n \times n$ matrix A is the sum of its diagonal elements:

$$\operatorname{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

(a) If A and B are $n \times n$ matrices, prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. (Use the definition of matrix multiplication and summation notation in your proof.)

(b) If P is an invertible $n \times n$ matrix, prove that $\operatorname{tr}(PAP^{-1}) = \operatorname{tr}(A)$.

(c) Consider the following ordered basis for $\mathcal{M}_{2\times 2}(F)$:

$$\alpha = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle.$$

The trace defines a function $\operatorname{tr}\colon \mathcal{M}_{2\times 2}(F) \to F$, and it is not hard to check that it is a linear transformation of vector spaces over F (you don't have to prove that here, although it is a good exercise to do). Compute the matrix representing the trace function $\operatorname{tr}\colon \mathcal{M}_{2\times 2}(F) \to F$ with respect to α for the domain and with respect to the basis $\{1\}$ for the codomain.

Problem 2. Let

$$A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & -3 \\ 4 & 1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & 4 \\ -1 & -2 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}.$$

Compute, if possible, the following. If it is not possible, explain why.

(a) AB,

(b) $A(2B + 3C)$,

(c) $(AB)D$,

(d) $A(BD)$,

(e) AD.

Problem 3. Let V be a vector space over a field F. Recall that the identity function $\mathrm{id}_V\colon V \to V$ is given by $\mathrm{id}_V(v) = v$ for all $v \in V$. This function is linear (if you are not convinced, prove it, but you do not have to turn in that proof).

(a) Let V be a vector space of dimension n and let $\mathcal{B}$ be an ordered basis for V. Show that the matrix representing id_V with respect to the basis $\mathcal{B}$ for both the domain and the codomain is I_n (the $n \times n$ identity matrix).

(b) Let V and W be vector spaces of dimension n and let $f\colon V \to W$ be an isomorphism with inverse $f^{-1}\colon W \to V$. Let $\mathcal{B}$ and $\mathcal{D}$ be ordered bases for V and W, respectively. If A is the matrix representing f with respect to the bases $\mathcal{B}$ and $\mathcal{D}$, what is the matrix for f^{-1} with respect to the bases $\mathcal{D}$ and $\mathcal{B}$? Justify your answer.

(c) Consider the linear transformation $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (3x + y, -x + 4y)$. Using part (b), find the inverse of f.